Aspects of twistors and tractors in field theory

Jack Williams

Department of Applied Maths and Theoretical Physics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

Clare College

April 2021

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Jack Williams
April 2021

Acknowledgements

I am extremely grateful to my supervisor David Skinner for his support and guidance throughout the past four years. His inexhaustible supply of ideas, guided by extraordinary intuitive understanding of physics, has been a constant source of inspiration for our work together. I want to thank him particularly for the generosity he has shown me with his time, and for spending more than a few long evenings puzzling over a blackboard.

Early in my PhD, I had the pleasure of collaborating with Tim Adamo, whose sense of humour enlivened even the longest and most laborious calculations. His patience and willingness to explain things in simple terms made for a smooth start to my PhD.

My officemates, Roland, Antoni, Carl and James have provided much needed, and occasionally mathematical, distractions as well as moral support and cake. Long lunches have been excellent respite, and I would like to thank Alice, Charles, Chris, Giuseppe, Julija, Manda, Nathan, Roxana and Toby in particular. Outside of physics, I feel fortunate to have the friendship of many kind and supportive people. I am especially thankful to Sarah, Anna, Pavel, Megan, Jesse, Adrien, Ellie and Johanna. Most of all, I would like to express my gratitude to Jess, who was by my side through the arduous route to this thesis, especially the latter stages.

Finally, I am grateful for my family, for their immense support and encouragement at every step.

Abstract

This thesis is divided into three sections.

The first begins with a review of the Fefferman-Graham embedding space construction and the closely-related tractor calculus, which provides a simple way to catalogue and invent conformally invariant operators. In particular, the conformal wave operator arises as the descendant of the ordinary wave operator on the Fefferman-Graham space. We construct a worldline action on the Fefferman-Graham space whose equations of motion descend to a conformally coupled scalar on the base manifold. We present also a novel Fefferman-Graham worldline action for the massless Dirac equation and demonstrate that it descents to a Dirac spinor on the base manifold. Unlike the scalar, the massless Dirac equation is conformally invariant without modification.

The second concerns the application of twistor theory to five-dimensional anti-de Sitter space. The twistor space of AdS_5 is the same as the ambitwistor space of the four-dimensional conformal boundary; the geometry of this correspondence is reviewed for both the bulk and boundary. A Penrose transform allows us to describe free bulk fields, with or without mass, in terms of data on twistor space. Explicit representatives for the bulk-to-boundary propagators of scalars and spinors are constructed, along with twistor action functionals for the free theories. Evaluating these twistor actions on bulk-to-boundary propagators is shown to produce the correct two-point functions.

In the final chapter, we construct a minitwistor action for Yang–Mills–Higgs theory in three dimensions. The Feynman diagrams of this action will construct perturbation theory around solutions of the Bogomolny equations in much the same way that MHV diagrams describe perturbation theory around the self–dual Yang Mills equations in four dimensions. We also provide a new formula for all tree amplitudes in YMH theory (and its maximally supersymmetric extension) in terms of degree d maps to minitwistor space. We demonstrate its relationship to the RSVW formula in four dimensions and show that it generates the correct MHV amplitudes at $d = 1$ and factorizes correctly in all channels for all degrees.

Table of contents

Chapter 1

Background

The causal structure of spacetime is of fundamental importance to physics. Information cannot propagate between points which are spacelike-separated, whereas an adventurous observer may travel between any two timelike-separated points. It is this fact which allows us to do physics at all: experiments conducted on Earth can remain free of interference from distant events unseen to the experimentalist, however extreme they may be.

Central to the causal structure of spacetime are null cones, which divide space into three regions: those in our timelike past, from which we can receive information, those in our timelike future, which can be influenced by our actions, and those outside our light cone, with which we cannot interact. Even more fundamental are null vectors, which show the directions along which light rays can travel away from a particular point. The set of directions in which null rays can travel away from a source is indexed by S^2, one for each point in the sky. A more sophisticated way to see this is to view the 3-dimensional light cone in $\mathbb{R}^{1,3}$ as a bundle over S^2 whose fibres are the null generators. The Lorentz action of $SO(1,3)$ on Minkowski space preserves the lightcone and descends to an action on the space of null rays through the origin, and hence also to a group action on S^2.

One can ask how the positions of stars in the sky move under a Lorentz boost applied to the Earth. It is easily checked that while the stars do move, the cross ratio of any set of four holomorphic positions on S^2 is an invariant. Cross ratios are preserved only by Möbius transformations, so this must be the effect of a boost on the set of null directions. This is the key insight: the celestial sphere S^2 can be given a unique complex structure and identified as the Riemann sphere. Relativity reveals a natural complex structure in the sky invisible to the Galilean observer.

1.1 Twistor space for flat Minkowski spacetime

This introduction to twistor theory summarises well-known work; see for example [85, 63, 110, 12].

A point x^μ in space time can be encoded as the 2×2 complex Hermitian matrix as

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + i x^2 \\ x^1 - i x^2 & x^0 - x^3 \end{pmatrix} = x^\mu \sigma_\mu^{\alpha\dot{\alpha}}, \tag{1.1}$$

where $\sigma_\mu^{\alpha\dot{\alpha}}$ are the Pauli matrices. The determinant of this matrix is the Lorentzian proper length

$$\det x = \frac{1}{2}\left[\left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 \right], \tag{1.2}$$

which is invariant under the action of $SO(1,3)$. This determinant is preserved by the action of $SL(2,\mathbb{C})$ by conjugation $x^{\alpha\dot{\alpha}} \mapsto \bar{x}^{\alpha\dot{\alpha}} = t^\alpha_{\ \beta} x^{\beta\dot{\beta}} \bar{t}^{\dot{\alpha}}_{\ \dot{\beta}}$, and so this action must move the corresponding spacetime vector according to a Lorentz transformation $x^\mu \mapsto \bar{x}^\mu = \Lambda^\mu_{\ \nu} x^\nu$. We have constructed a map $SL(2,\mathbb{C}) \to SO(1,3)$ with kernel $\{\pm I\}$ and the Lorentz action on null rays translates to an $SL(2,\mathbb{C})$ action on Hermitian matrices.

The matrix $x^{\alpha\dot{\alpha}}$ corresponds to a null vector precisely when its determinant vanishes. When this happens, its image and kernel as a map from $\mathbb{C}^2$ to $\mathbb{C}^2$ must have dimension one. A natural invariant object of the entire null ray $\left\{k x^{\alpha\dot{\alpha}} | k \in \mathbb{R}\right\}$ is the kernel. This kernel is spanned by a single two-component complex vector and can be identified with a unique point λ_α in $\mathbb{CP}^1$. Points on the Riemann sphere $\mathbb{CP}^1$ parametrise light rays through the origin. The $SO(1,3)$ action on null rays induces an $SL(2,\mathbb{C})$ action on $\mathbb{CP}^1$, which gives the expected Möbius maps. $\mathbb{CP}^1$ is diffeomorphic to the Euclidean 2-sphere, so this was already clear from a real geometric perspective. But the extra complex structure illuminates the action of $SO(1,3)$ by Möbius maps.

The matrices $x^{\alpha\dot{\alpha}}$ map points in $\lambda_\alpha \in \mathbb{CP}^1$ according to

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha, \tag{1.3}$$

which describes a line in the space $\mathbb{CP}^3$ with homogeneous coordinates $Z = (\lambda_\alpha, \mu^{\dot{\alpha}})$. We exclude the 'line at infinity', $I = \{\lambda_\alpha = 0\}$. The remaining $\mathbb{CP}^3 \setminus I$ is called the *twistor space*. This gives a complementary point of view, in which spacetime points correspond to lines in twistor space. This essential feature which will allow the construction Penrose transform in the next section.

From the perspective of null rays, one can fix a point $Z = (\lambda_\alpha, \mu^{\dot\alpha})$ in twistor space an ask which spacetime points correspond to lines passing through that point. If $\mu^{\dot\alpha} = 0$, we already know the answer: x must lie on the null ray through the origin corresponding to λ_α. If $\mu^{\dot\alpha} \neq 0$, then if we can find one solution to equation (1.3), all others must differ from it by points whose kernel is λ_α. Again, we obtain a null ray, but offset from the origin.

This gives a point-line duality: points in spacetime correspond to holomorphic lines in twistor space. The λ_α components hold information about the direction of our null ray, and the $\mu^{\dot\alpha}$ components hold information about the location of the ray relative to the origin.

1.2 The Penrose transform for flat Minkowski spacetime

The wave equation is one of the simplest and most important partial differential equations. In this section, we explain how its solutions can be represented by functions on twistor space, and describe the Penrose transform as a map between physical solutions and twistorial representatives.

Given a $(0,1)$-form $f(\lambda,\mu)$ on the twistor space of complexified Minkowski space $\mathbb{CP}^3 \setminus I$, one can construct a function $\phi : \mathbb{R}^{1,3} \to \mathbb{C}$ on spacetime in a canonical way

$$\phi(x) = \int_{L_x} f(\lambda,\mu)\langle\lambda d\lambda\rangle, \tag{1.4}$$

where L_x is the line in twistor space corresponding to spacetime point x and $\langle\lambda d\lambda\rangle$ is the projective measure $\epsilon^{\alpha\beta}\lambda_\alpha d\lambda_\beta$. The 1-form $f(\lambda,\mu)$ must have homogeneity -2 so that the integral is well defined. In practice, the integral over L_x means setting $\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$ and performing the integral over λ. This is the *Penrose transform.* It can be seen to satisfy the spacetime wave equation,

$$\begin{aligned}
\Box\phi(x) &= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\phi(x) && (1.5)\\
&= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\int_{L_x} f(\lambda_\alpha, x^{\alpha\dot\alpha}\lambda_\alpha)\langle\lambda d\lambda\rangle && (1.6)\\
&= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\int_{L_x}\lambda_\alpha\lambda_\beta\frac{\partial}{\partial\mu^{\dot\alpha}}\frac{\partial}{\partial\mu^{\dot\beta}} f(\lambda_\alpha, x^{\alpha\dot\alpha}\lambda_\alpha)\langle\lambda d\lambda\rangle && (1.7)\\
&= 0, && (1.8)
\end{aligned}$$

provided f depends on the twistor variables only holomorphically ($\bar\partial f = 0$). Moreover, any solution to the wave equation can be obtained this way [48].

It is easy to see that the Green's function, $\phi = \frac{1}{(x-y)^2}$, can be written as a Penrose transform. Two twistor points A, B determine a twistor line, and hence a spacetime point y. A simple integration shows that the Penrose representative

$$f(Z) = \frac{1}{A \cdot Z} \bar{\partial} \left(\frac{1}{B \cdot Z} \right) \tag{1.9}$$

reproduces the Green's function on spacetime.

The map between solutions and their Penrose representatives is not one-to-one. Indeed, any $\bar{\partial}$-exact form may be added to a Penrose representative $f \mapsto f + \bar{\partial} g$ without changing the value of the integral. Representatives which differ by exact derivatives should be regarded as equivalent, and the map is really between cohomology classes of $\bar{\partial}$, $\bar{\partial}$-closed forms on twistor space modulo $\bar{\partial}$-exact forms. We will also use Penrose transforms for fields of other spin throughout this thesis. Particles of spin s are represented by elements of the cohomology class $H^{0,1}\left(\mathbb{CP}^3 \setminus I, \mathcal{O}(-2s-2)\right)$ [83, 48].

1.3 Summary of thesis

Twistor theory has been highly successful for tackling problems of classical geometry. The utility of the Penrose transform to reformulate not just linear problems, such as the wave equation, but also non-linear ones, such as the self-dual Yang-Mills equations and self-dual gravity, in terms of simpler, unconstrained problems on twistor space is among its most important achievements. Solutions to complicated geometric problems in spacetime are exchanged for holomorphic functions in twistor space.

It is natural to ask whether this classical geometric success of twistor theory can be matched by applications in quantum field theory.

1.3.1 Twistors for conformally invariant theories

The ambient metric construction of Fefferman and Graham [51, 56] is a powerful tool for studying conformal geometry. It allows one to build, from a given n dimensional manifold M, a manifold $\mathcal{M}$ of dimension $n+2$ which depends only on the conformal class of the metric on M, and not on the particular representative. The manifold $\mathcal{M}$ and all objects constructed from it are necessarily conformal invariants, meaning that they are preserved by Weyl transformations of the metric $g \mapsto \Omega^2 g$. A null cone inside $\mathcal{M}$ contains as its sections embeddings of M and all conformally related manifolds.

The metric on $\mathscr{M}$ is chosen to be Ricci flat by solving a recurrence relation order-by-order in a parameter in a direction off the cone. Whether this construction is possible depends of the vanishing of a certain obstruction tensor, which in four dimensions is the conformally invariant Bach tensor.

The primary motivation for this construction was to discover and classify conformally invariant tensors beyond a handful which were already known such as the Weyl tensor and the Bach tensor. This aim was achieved [51, 50]. Among the conformal tensors discovered are the various obstruction tensors arising in the construction of the Fefferman-Graham ambient metric.

Our purpose for the ambient construction is different. In chapter 1, we describe a worldline quantum mechanical theory whose target space is the Fefferman-Graham space for a given base manifold M. In doing so, we ensure that the theory we construct is conformally invariant. By imposing constraints, we reduce the Hilbert space to lower dimensional one and the theory descends to a conformally invariant one on M. In general, one encounters ordering ambiguities when quantising quantum theories, but we demonstrate that no such ambiguities arise in our theory on M by virtue of its construction from the ambient space.

Our goal is to understand quantum fields in the presence of general conformal curvature. We consider both a scalar, which descends to a conformally coupled scalar, and a Dirac spinor, which descends to a ordinary Dirac spinor on M. Both these descended theories are conformally invariant as desired. We do so first in the simple case of a conformally flat base manifold, whose ambient space is flat Minkowski space, but then are able to generalise to the case of conformal curvature without significant additional conceptual difficulty.

The Fefferman-Graham construction has also found applications in the study of the AdS/CFT correspondence. AdS space of dimension $n+1$ is viewed as the projectivisation of the Fefferman-Graham embedding space of dimension n, and asymptotically AdS spaces can be constructed from fluctuations on the conformal boundary [94].

1.3.2 Twistors and the AdS/CFT correspondence

The holographic principle has its origins in black hole thermodynamics. While the entropy of most systems grows as the cube of its length scale, a black hole's grows only quadratically in its radius, proportional to the area of its event horizon. This suggests that the only degrees of freedom are on the surface, and can be encoded in a space of one fewer dimension that the space in which the black hole lives. More recently, Maldacena conjectured a duality between type IIB string on $\mathrm{AdS}_5 \times S^5$ space, and the conformal field theory $\mathcal{N} = 4$ super Yang-Mills on its conformal boundary [73, 114]. Much evidence has been found in favour of this conjecture (e.g [39]), and it is widely expected to be true. Nonetheless, many important questions remain unanswered. Self-dual $\mathcal{N} = 4$ super Yang-Mills is a simplification of the well-known example which retains the conformal invariance of the original. This theory should be dual to a simpler string theory in the bulk, providing a simpler example of an AdS/CFT correspondence than is currently available. One might hope that this setting would permit a more secure understanding of the correspondence than is possible in the full duality of $\mathcal{N} = 4$ super-Yang-Mills.

In this thesis, we look only at the basic steps of finding twistor representatives for scalar and spinor bulk-to-boundary propagators, which will be crucial in any further study of the subject. By writing some of the ingredients of the AdS/CFT correspondence in twistor space, one might hope to find a theory on twistor space which generates the observables.

In chapter 3, we describe the twistor space of AdS_5 and construct explicit twistor representatives for scalar and spinor bulk-to-boundary propagators, some of the most fundamental objects in the correspondence, and verify that a natural twistor action for these fields reproduces the expected form of the two-point boundary correlation function.

1.3.3 Twistors for amplitudes in quantum field theory

A major success of twistor theory is the RSVW formula [115, 89] for all tree amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. The amplitudes generated by this functional are organised around the self–dual sector, and indexed by a parameter d grading the NMHV degree of the scattered particles.

$$\sum_{d=0}^{\infty} g_{(4)}^{2d} \int \mathrm{d}\tilde{\mu}_{\tilde{C}}^{(d)} \log\det(\bar{\partial} + \mathscr{A}_4)|_{\tilde{C}} \tag{1.10}$$

where the integral is taken over all degree d holomorphic maps $Z : \mathbb{CP}^1 \to \mathbb{CP}^{3|4}$ from a rational curve to $\mathcal{N} = 4$ twistor space $\mathbb{CP}^{3|4}$. Expanding in powers of the on-shell background field $\mathscr{A}_4$, this formula is a generating functional for all tree amplitudes in $\mathcal{N} = 4$ SYM_4. Each of the elements of this formula will be explained in the context of three-dimensional Yang-Mills-Higgs theory in chapter 3. The compactness of this generating functional, compared with other methods of calculation, such as the Feynman diagram expansion, is its most impressive feature.

The proof that this functional produces the correct amplitudes proceeds by checking first that it produces the correct 3-point functions, and then that the residues of its poles factorise correctly, so that it satisfies the BCFW recursion relations expected of an amplitude [103, 95, 25, 24].

In chapter 3, we present a dimensional reduction of this theory, the three dimensional Yang-Mills-Higgs theory, and offer an analogous formula for the full set of tree-level amplitudes. We demonstrate its correctness by calculating 3-point functions and checking the factorisation property.

Chapter 2

Tractors for conformal worldline theories

2.1 Introduction

This section begins with an introduction to the conformal geometry required to construct a conformal worldline quantum mechanical model. The conformally flat sphere is the simplest case and is discussed first, before generalising to the general case of conformal curvature and arbitrary topology. The basic idea, described in [51], is to construct from our base manifold (M, g) a higher dimensional spacetime $(\mathscr{M}, \mathscr{G})$ by adding one spacelike and one timelike direction. The higher dimensional spacetime depends only on the conformal class of the base metric g, though this fact will not be manifest, and inside it lies a null cone $\mathscr{N}$ whose sections (not necessarily planar) are embeddings of (M, g) and all spacetimes which can be obtained from (M, g) by Weyl rescalings of the metric. The direction along the null generators of the cone corresponds to Weyl rescaling.

In the simple case of a conformally flat base, the ambient space is flat $\mathbb{R}^{1,n+1}$ and the embeddings of spacetimes conformally equivalent to the sphere arise as sections of the light cone at the origin. The 'horizontal section' obtained by intersection with a horizontal plane induces a round sphere, while the parabolic section induces flat $\mathbb{R}^n$. Hyperbolic space arises from hyperbolic sections of the cone.

On restriction to these 'conic sections', operators on the ambient space descend to operators on the base manifold. Given that the ambient space depends only on the conformal class of the base manifold, these operators must be conformal invariants.

We consider actions for worldline quantum mechanical theories in the ambient space. By imposing natural constraints, we restrict these theories to the cone and a homogeneity along the null generators of the cone, which corresponds to conformal weight of various fields. We treat only the massless scalar and Dirac spinor, but other possibilities exist. The massless free scalar action descends to a scalar on the base whose equation of motion is the conformal wave equation. Since the massless Dirac equation is conformally invariant, we find the descendent of a massless Dirac spinor in the ambient space to be a massless Dirac spinor on the base.

Ambiguities typically arise in the quantisation of quantum mechanical worldline theories. For example, ordering ambiguities may arise in the Laplacian or in other natural operator expressions. One solution, is to impose a particular renormalisation scheme. We will see that important properties of the ambient space construction allow us to avoid these problems and construct a worldline action in a natural and conformally invariant manner.

2.2 Conformal geometry

2.2.1 The conformal sphere as a lightcone inside $\mathbb{R}^{1,n+1}$

The n-sphere, defined as the subset of $\mathbb{R}^{n+1}$, can be viewed as the horizontal section of a (light)cone in $\mathbb{R}^{1,n+1}$, where the round metric is induced on it by the ambient space $\mathbb{R}^{1,n+1}$. In this example, the sphere is our base manifold M and the flat Lorentzian $\mathbb{R}^{1,n+1}$ in which it is embedded is our ambient space $\mathscr{M}$. Alternatively, one could take the parabolic section of the same cone by intersecting with a plane parallel to the edge of the cone. The resulting induced space is flat $\mathbb{R}^n$. It is worth noticing that the round n-sphere and flat $\mathbb{R}^n$ are conformally equivalent, the relation being given by stereographic projection.

An embedding of a manifold M into the ambient space takes the form of a map $\phi : M \to \mathscr{M}$ and is represented in coordinates as

$$\phi(x) = X^\mu(x) \qquad \text{where} \qquad \eta_{\mu\nu} X^\mu X^\nu = 0$$

A conformal (Weyl) transformation of the base metric is specified by a positive function $\Omega : \mathscr{M} \to \mathbb{R}_+$ which is used to transform the metric to a conformal relative $\tilde{g} = \Omega^2 g$. This information can be recorded on our cone by stretching the section to

$$\tilde{\phi}(x) = \Omega(x) X^\mu(x)$$

and the constraint $\eta_{\mu\nu} X^\mu X^\nu = 0$ clearly still holds.

This encoding of the conformal transformation onto the cone may seem artificial, but to see that it is deeper, we can calculate the metric induced on a general curved section. A suitable parametrisation is

$$X^\mu = \frac{\Omega(x)}{1+\mathbf{x}^2}\begin{pmatrix} 1+\mathbf{x}^2 \\ 1-\mathbf{x}^2 \\ 2\mathbf{x} \end{pmatrix}, \tag{2.1}$$

since this satisfies $\eta_{\mu\nu}X^\mu X^\nu = 0$ and the Euclidean distance to the origin is $\sqrt{2}\,\Omega(x)$. It is easy to check that the induced metric is

$$ds^2 = \eta_{\mu\nu}dX^\mu dX^\nu = \Omega(x)^2\frac{4d\mathbf{x}^2}{(1+\mathbf{x}^2)^2}, \tag{2.2}$$

which is related to the round metric on the sphere $\frac{4d\mathbf{x}^2}{(1+\mathbf{x}^2)^2}$ by the conformal factor $\Omega(x)$. The simplest example is the round sphere given by $\Omega(x) = 1$, which lies in the horizontal plane $X^0 = 1$. The flat plane arises as the case $\Omega(x) = \frac{1}{2}\left(1+\mathbf{x}^2\right)$ and lies inside the plane $X^0 + X^1 = 1$, which is indeed parallel to the side of the cone.

A particularly natural but small collection of sections are those which arise as intersections with a plane: ellipses, parabolas and hyperbolas. The ellipses all produce spheres whose radii increase with eccentricity, before deforming smoothly into the flat plane in the parabolic limit. Beyond the parabola, the hyperbolic sections occupy only a bounded portion of the **x**-plane and accordingly produce the bounded hyperbolic disc. In fact, these spaces are distinguished intrinsically, being the only conformally flat metrics which are also Einstein.

This follows because the vacuum Einstein equation for the induced manifold is

$$\partial_\mu\partial_\nu\Omega - \frac{1}{n}\partial^2\Omega\delta_{\mu\nu} = 0, \tag{2.3}$$

whose solutions are $\Omega = a + 2b_\mu X^\mu + cX^2$. This is satisfied by points on the plane

$$\begin{pmatrix} \frac{1}{2}(a+c) \\ \frac{1}{2}(a-c) \\ \mathbf{b} \end{pmatrix}\cdot X = 0. \tag{2.4}$$

2.2.2 Homogeneous functions on M as functions on $\mathcal{N}$

A simple but useful observation is that a function $f: M \to \mathbb{C}$ of weight λ which depends on the conformal scale can be viewed as a function on the null cone $\mathcal{N}$, with dependence on the scale replaced by dependence on the position along the null generator of the cone. In particular, a homogeneous function $f : M \to \mathbb{C}$ can be viewed as a function with homogeneity in X, since scaling X is equivalent to sliding up the side of the cone. Homogeneity of weight λ translates to the condition $X \cdot \partial f = \lambda f$.

2.2.3 The tractor bundle

One of the early motivations for the tractor bundle construction was to proliferate and systematically catalogue conformal invariants beyond the few that were already known – the Weyl tensor and the Bach tensor [97]. The idea is to take a vector field in the ambient space $\mathscr{M}$ which is restricted (but not necessarily tangent) to the null cone $\mathscr{N}$ and is conformally invariant, or homogeneous of degree zero.

A *tractor* will be defined to be a vector field which is homogeneous of degree zero and parallel transported along a single null generator of the cone ([38], following [98–100]). The vector belongs to the tangent space to the entire ambient space $\mathscr{M}$ and may have components pointing off the cone. A *tractor field* consists of a tractor on each null generator of the cone and descends to a field on the base manifold M. It is clear that the space of tractors at a point in M is a vector space and so the tangent space to the ambient manifold descends to a vector bundle over M, the *tractor bundle* $\mathscr{T}$. Tractor fields are sections of the tractor bundle. We will not discuss the construction of conformal invariants using tractors, but details are given in [38].

More important for the present discussion will be the connection induced on the tractor bundle by the ambient space and the question of the existence of flat sections. In this example it is clear that we have $n+2$ independent flat sections which descend from flat vector fields on $\mathbb{R}^{n+2}$. This fact may not hold in the general case. In fact, we will see that the existence of a flat section of the tractor bundle is equivalent to the existence of an Einstein section of the cone.

2.2.4 The tractor connection

As a first step towards constructing the ambient space for a general base manifold and metric, we give an intrinsic description of the tractor connection on $\mathscr{T}$. There are two main advantages of this intrinsic construction. The first is that dependence on only the conformal class of the base metric is manifest. The second is that it makes clear why parallel sections of $\mathscr{T}$ correspond to Einstein metrics in the class. This section closely follows the presentation of [38].

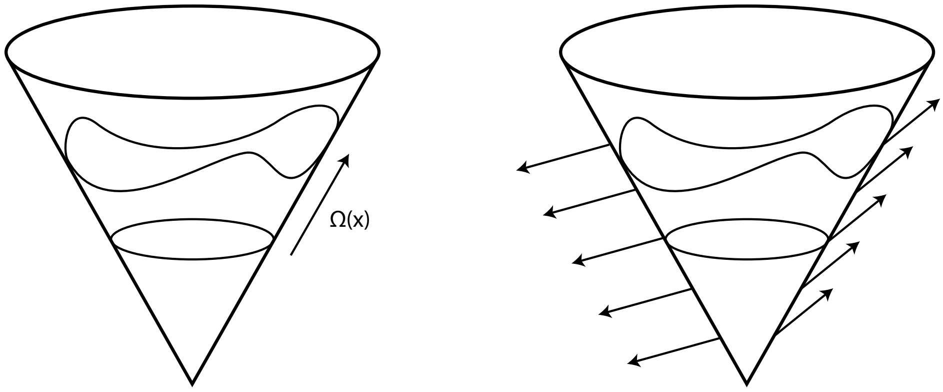

Fig. 2.1 Sections of the null cone in Lorentzian $\mathbb{R}^{n+2}$ corresponding to the round sphere and a stretched section, from which we can read off the scale factor Ω relating the two. The picture on the right shows two tractors at different points on M.

Since we anticipate an $(n+2)$-dimensional embedding space, it is natural that the tractor bundle will be an $(n+2)$-dimensional vector bundle over M. To specify the bundle, it is necessary to describe trivialising patches on M the $GL(n+2)$ action relating the components of vectors on the patch overlaps. We will take the patches to be the usual coordinate patches on M and the components of the vector to be an n-vector μ^i transforming in the usual way together with two additional scalars, ν and σ. The scalars will have weights -1 and +1 and the tractor bundle will be defined as $\mathscr{E}(1) \oplus \mathscr{E}_i(1) \oplus \mathscr{E}(-1)$, where $\mathscr{E}(w)$ is the bundle of conformal densities of weight w and $\mathscr{E}_i(w) = \mathscr{E}(w) \otimes T^*M$.

Before explicitly describing the connection, it will be useful to find a simple condition for a metric to be conformal to an Einstein metric. Such metrics are said to be *almost Einstein*.

It is well known [23, 16, 49, 117, 42, 102] that the conformal Laplace equation

$$\nabla^2 f - \frac{n-2}{4(n-1)} R f = 0, \tag{2.5}$$

where R is the Ricci scalar, is conformally invariant provided f has homogeneity $\frac{2-n}{2}$. This equation is the trace of

$$\nabla_i \nabla_j f - \frac{n-2}{2} P_{ij} f = 0, \tag{2.6}$$

where $P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)} g_{ij}\right)$ is the Schouten tensor. Less well known is that a similar equation

$$\nabla_{(i} \nabla_{j)_0} \sigma + P_{(ij)_0} \sigma = 0, \tag{2.7}$$

where $T_{(ij)_0}$ denotes the symmetric traceless part of T_{ij}, is conformally invariant provided σ has homogeneity 1 [13]. The traceless part of the Schouten tensor is proportional to the traceless part of the Ricci (and hence Einstein) tensor. If g_{ij} is Einstein, this must vanish and the equation reduces to $\nabla_{(i}\nabla_{j)_0}\sigma = 0$, which is evidently satisfied by $\sigma = 1$. If we make a conformal transformation from an Einstein metric to $\Omega^2 g_{ij}$, then σ transforms to $\sigma = \Omega$ and (2.7) continues to hold by conformal invariance.

Conversely, if we have a positive solution $\sigma = \Omega$ to (2.7), then we can perform the inverse conformal transformation Ω^{-1}. Under this transformation, σ transforms back to $\sigma = 1$ and satisfies (2.7) with respect to the new metric. Since the first term vanishes, the second must also and we learn that the traceless Schouten tensor vanishes. Since the Schouten tensor is a trace-adjusted Ricci tensor, it follows that the traceless Ricci tensor vanishes too and the new metric is Einstein. Therefore, the existence of an Einstein metric in the conformal class is equivalent to the existence of a positive solution to (2.7). For this reason, (2.7) is called the *almost Einstein equation.*

To understand this equation better, notice that solving the almost Einstein equation is obviously equivalent to solving

$$\nabla_i\nabla_j\sigma + P_{ij}\sigma - \nu g_{ij} = 0.$$

If we set $\mu_i = \nabla_i\sigma$, then this is replaced by a system of two equations

$$\begin{aligned} \nabla_i\sigma - \mu_i &= 0, && (2.8)\\ \nabla_i\mu_j + P_{ij}\sigma - \nu g_{ij} &= 0. && (2.9) \end{aligned}$$

A simple calculation shows that these imply a third equation,

$$\nabla_i\nu - P_{ij}\mu^j = 0, \tag{2.10}$$

where $\mu^j = g^{ji}\mu_i$.

Now we are ready to see how flat sections of the tractor bundle correspond to Einstein metrics. We simply define the tractor bundle to have components (ν, μ_i, σ) and the tractor connection to be

$$\nabla_i^{\mathcal{T}} \begin{pmatrix} \nu \\ \mu^j \\ \sigma \end{pmatrix} = \begin{pmatrix} \nabla_i\nu - P_{ij}\mu^j \\ \nabla_i\mu^j + P_i^j\sigma - \nu\delta_i^j \\ \nabla_i\sigma - \mu_i \end{pmatrix}. \tag{2.11}$$

From a parallel section of this connection, we can read off the conformal factor σ satisfying the almost Einstein equation and, conversely, we can construct a flat section from a given almost Einstein solution σ.

This definition of the tractor connection looks as though it depends on the conformal factor. However, two tractor bundles constructed using two different, but conformally equivalent, metrics are isomorphic. Suppose g_{ij} is a fixed representative of a particular conformal class. The isomorphism relating the bundle constructed using $e^{2u(x)}g_{ij}$ to the one constructed from $e^{2\tilde{u}(x)}g_{ij}$ is given by

$$\begin{pmatrix} e^{-\tilde{u}}\tilde{\nu} \\ e^{\tilde{u}}\tilde{\mu}^i \\ e^{\tilde{u}}\tilde{\sigma} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\nabla^i u & \delta^i_j & 0 \\ \frac{1}{2}(\nabla u)^2 & -\nabla_j u & 1 \end{pmatrix} \begin{pmatrix} e^{-u}\nu \\ e^{u}\mu^j \\ e^{u}\sigma \end{pmatrix}. \tag{2.12}$$

It is important that the tractor connection be covariant with respect to this isomorphism [38], a fact which can be checked directly but which will follow immediately once we see how the connection arises from the ambient space construction.

2.2.5 The general ambient space construction

The crucial step is the generalisation of the ambient space construction to n-dimensional base manifolds (M, g) which are not conformally flat. A natural guess would be to start with a cone whose horizontal sections are isometric to constant conformal scalings of (M, g), and then extend the metric off the cone such that the ambient metric is flat. Unfortunately, this is not possible.

A more realistic suggestion is to ask that the ambient space be Ricci-flat. It can be shown that such a metric must take the form

$$\mathcal{G} = -2\rho dt^2 - 2t dt d\rho + t^2 g_{ij}(x,\rho) dx^i dx^j, \tag{2.13}$$

where $g_{ij}(x,0) = g_{ij}(x)$ is the base metric and the higher order terms in ρ are chosen order-by-order such that $\mathcal{G}$ is Ricci-flat [51]. This is always possible to order $\frac{n}{2}$ in ρ, and the remainder of the series is uniquely determined provided a certain obstruction tensor, which will be described later vanishes. The cone is the surface $\rho = 0$ and t is a scaling variable (the metric is homogeneous of degree 2 in t). The section of the cone $t = 1$ has induced metric $g_{ij}(x)$ so this is where we find the embedding of our base manifold.

In the special case where $g_{ij}(x) = e^{2u(x)}\delta_{ij}$ is conformally flat, the function $g_{ij}(x,\rho)$ terminates at quadratic order and we have [51, 47]

$$g_{ij}(x,\rho) = g_{ij} - 2\rho P_{ij} + \rho^2 g^{kl} P_{ik} P_{jl}, \tag{2.14}$$

where $P_{ij}(x)$ is the Schouten tensor corresponding to $g_{ij}(x)$.

We expect this to reduce to the flat $\mathbb{R}^{n+2}$ considered in the previous section, and upon making the change of variables

$$\begin{aligned}
X^0 &= te^u\left[\frac{\mathbf{x}^2+1}{2} + e^{-2u}\rho\left(\frac{1}{2}(\nabla u)^2\,\frac{\mathbf{x}^2+1}{2} + \mathbf{x}\cdot\nabla u + 1\right)\right],\\
X^1 &= te^u\left[\frac{\mathbf{x}^2-1}{2} + e^{-2u}\rho\left(\frac{1}{2}(\nabla u)^2\,\frac{\mathbf{x}^2-1}{2} + \mathbf{x}\cdot\nabla u + 1\right)\right],\\
\boldsymbol{X} &= te^u\left[\mathbf{x} + e^{-2u}\rho\left(\frac{1}{2}(\nabla u)^2\,\mathbf{x} + \nabla u\right)\right],
\end{aligned} \tag{2.15}$$

it is seen that the ambient metric is indeed flat. We discovered this change of variables by finding the Killing forms of the Fefferman-Graham embedding space. Our parametrisation of conic sections (2.1) is now simply a special case of this transformation with $\rho = 0$ and $t = 1$ and yields the submanifold $(M, e^{2u}\delta_{ij})$, as expected. Another easy calculation finds that $X^2 = -2\rho t^2$ so $\rho = 0$ does describe the null cone.

Notice that in this case, the metric (2.13) appears to depend on g_{ij} but is in fact flat for all choices of scale $u(x)$. Hence, the ambient space depends only on the conformal class to which g_{ij} belongs, in this case the class of conformally flat metrics.

The tractor connection is simply the Levi-Civita connection of the ambient space [38]. It is easy to explicitly recover the intrinsic definition (2.11) by restricting the components of the connection to a particular section of cone $\rho = 0$, $t = 1$ corresponding to some choice of conformal scale. Although the resulting bundle appears to depend on this section, we saw earlier that different sections produce isomorphic bundles. How should bundles over different sections be identified in the extrinsic picture?

In the conformally flat case, we defined a tractor at a point in M to be a vector field of homogeneity zero along a null generator of the cone. This allows us to identify vectors living at different sections of the cone (but on the same null ray) and hence to identify the different tractor bundles. The isomorphism (2.12) gave an intrinsic relation between the components of vectors in two bundles corresponding to two different sections.

We can now understand this isomorphism extrinsically. The components of corresponding vectors in the coordinate basis inherited from the flat ambient coordinates (2.15) should be equal. So we can transform the components of each vector to the flat coordinate basis and equate them. This will give us a relation between the components of vectors in the more natural bundle coordinates (v, μ_i, σ). Explicit calculation yields the same matrix as (2.12).

Following this example in the general case, we define a tractor to be a vector field along a generator of the cone which is parallel with respect to the ambient metric. This gives us a bijection between bundles constructed from different sections: corresponding vectors are obtained by parallel transport along null generators of the cone.

We have now constructed both the ambient space and tractor bundle in a way which superficially used the base metric, but the result turned out to depend only on the conformal class. These constructions will be useful for building a conformally invariant worldline action.

2.3 Worldline scalar actions

When constructing worldline actions on the base manifold, the constraints such as P^2 must be quantised. Ordering ambiguities may be partially resolved by requiring that the quantum operator be diffeomorphism invariant [18]. Under such a scheme, the wave operator would take the form $\nabla^2 + kR$, where R is the Ricci curvature and k is an undetermined constant. A particularly attractive choice of k would be $-\frac{n-2}{4(n-1)}$, so that the wave operator becomes the conformal Laplacian (2.5), but such a choice is unmotivated.

The virtue of the Fefferman-Graham embedding space for constructing such actions is that it is Ricci flat, so there is a canonical choice of diffeomorphism-invariant quantisation. It is conformal, so we expect to obtain the conformal wave equation as the equation of motion upon descent to the base manifold. We show this to be correct, and demonstrate the wider applicability of actions descended from Fefferman-Graham space obtaining a Dirac spinor, which is conformal, from a Dirac action on the embedding space.

Some similar work has been done on the bosonic, conformally flat case in [21] and related papers.

2.3.1 The conformally flat case

The action for a free massless particle on the flat ambient $\mathbb{R}^{n+2}$ can be written in Hamiltonian form

$$S[X,P] = \int \left(\dot{X} \cdot P + \frac{\alpha}{2} P^2 \right) dt, \tag{2.16}$$

where α is a Lagrange multiplier [44]. Since we have seen that the null cone $X^2 = 0$ provides a model for the conformal geometry, we will restrict our model to the cone by imposing this an additional constraint. For canonical quantisation by Dirac's method [45, 40], the constraints should be first-class so we are forced to add a third constraint

$$\left\{\frac{1}{2}X^2, \frac{1}{2}P^2\right\} = \frac{1}{2}(X\cdot P + P\cdot X) = X\cdot P - i\frac{n+2}{2}.$$

The whole action becomes

$$S[X,P] = \int\left(\dot{X}\cdot P + \frac{\alpha}{2}P^2 + \frac{\beta}{2}X^2 + \frac{\gamma}{2}(X\cdot P + P\cdot X)\right)dt, \tag{2.17}$$

We impose these constraints directly on the Schrödinger wavefunction $\Psi(X)$ [69, 91, 41]. The cone constraint $X^2\Psi = 0$ implies that Ψ takes the form

$$\Psi(X) = \delta(X^2)\phi(X). \tag{2.18}$$

The scaling constraint $\left(-iX\cdot\partial - i\frac{n+2}{2}\right)\Psi = 0$ implies that

$$\begin{aligned}
0 &= \left(X\cdot\partial + \frac{n+2}{2}\right)\delta(X^2)\phi(X) &(2.19)\\
&= 2X^2\delta'(X^2)\phi(X) + \delta(X^2)X\cdot\partial\phi(X) + \frac{n+2}{2}\phi(X) &(2.20)\\
&= -2\delta(X^2)\phi(X) + \delta(X^2)X\cdot\partial\phi(X) + \frac{n+2}{2}\delta(X^2)\phi(X) &(2.21)\\
&= \delta(X^2)\left(X\cdot\partial + \frac{n-2}{2}\right)\phi(X) &(2.22)
\end{aligned}$$

and so ϕ has homogeneity $\frac{2-n}{2}$. The final constraint $\partial^2\Psi = 0$ descends to an equation on each section of the cone

$$\nabla^2\phi - \frac{n-2}{4(n-1)}R\phi = 0, \tag{2.23}$$

where R is the induced scalar curvature [38]. This will be shown directly in the next section in the more general case of a conformally curved base manifold. Here, we have obtained is the conformal Laplace equation (2.5) and ϕ has the correct homogeneity for conformal invariance. We have therefore succeeded in building a conformally invariant theory on M. This was to be expected because ∂ is the tractor connection on the ambient space, which depends only on the conformal class of the base manifold.

In addition the constraint equations, we need an inner product to fully define the Hilbert space. A natural Hermitian inner product on M is

$$\langle f,g\rangle = -i\int_\Sigma \sqrt{h}\, n^j\left(\bar{f}\nabla_j g - g\nabla_j \bar{f}\right), \tag{2.24}$$

where $\Sigma \subset M$ is a hypersurface, h is the induced metric and n^i is a unit normal. This inner product is a conformal invariant. Under a conformal transformation,

$$\langle f, g\rangle \mapsto -i\int_\Sigma \Omega^{n-1}\sqrt{h}\,\left(\Omega^{-1}n^j\right)\left(\left(\Omega^{\frac{2-n}{2}}\bar{f}\right)\nabla_j\left(\Omega^{\frac{2-n}{2}}g\right)-\left(\Omega^{\frac{2-n}{2}}g\right)\nabla_j\left(\Omega^{\frac{2-n}{2}}\bar{f}\right)\right) \tag{2.25}$$

$$= -i\int_\Sigma \sqrt{h}\,n^j\left(\bar{f}\nabla_j g+\frac{2-n}{n}\bar{f}g\partial_j(\log\Omega)-g\nabla_j\bar{f}-\frac{2-n}{n}\bar{f}g\partial_j(\log\Omega)\right) \tag{2.26}$$

$$= \langle f, g\rangle \tag{2.27}$$

Applying Stokes' theorem and using the conformal Laplace equation shows that this inner product is independent of Σ. To complete the discussion of the conformally flat case, we write this inner product in flat ambient coordinates

$$\langle f, g\rangle = -i\int_\Sigma \left(n\lrcorner D^{n+1}X\right)\delta(X^2)\frac{n\cdot\left(\bar{f}\partial g-g\partial\bar{f}\right)}{n^2}. \tag{2.28}$$

Here, $D^{n+1}X = \epsilon_{\mu_0\mu_1\cdots\mu_{n+1}}X^{\mu_0}dX^{\mu_1}\cdots dX^{\mu_{n+1}}$ is the projective measure on the ambient space and n is a lift of the normal n^i to the tangent space of the cone. The lift is not unique, so we will need to check that the resulting integrand is independent of the choice of lift. Two different lifts are related by $\tilde{n} = n + kX\cdot\partial$. The numerator $\tilde{n}\cdot\left(\bar{f}\partial g - g\partial\bar{f}\right)$ is preserved because f and g have the same weight. The denominator is preserved because n is tangent to the cone and $X^2 = 0$ on the support of the delta function. The measure is preserved because the extra contribution is proportional to $X\cdot\partial\lrcorner(X\cdot\partial\lrcorner \mathrm{d}^{n+2}X) = 0$.

2.3.2 The general curved case

It is now simple to generalise this worldline action to a general base manifold; the only difficulty is finding the correct way to generalise the constraints. A key observation is that the ambient metric (2.13) is quadratic in t, so the generator of dilations is $t\partial_t$. The vector field $X\cdot\partial$ generated dilations in the flat case so it is natural to try to identify these. Ordering ambiguities arise when quantising $P^2 = g^{\mu\nu}(X)P_\mu P_\nu$.

The constraints generalise to the curved case as follows. The X^2 constraint should be interpreted not as a coordinate expression in the special case of a flat embedding space, but as the metric length of the vector field generating scalings of this space. Since t is the coordinate along the cone measuring scaling, this vector field should be $X = t\partial_t$. Hence,

$$\frac{1}{2}X^2 \longrightarrow \frac{1}{2}(t\partial)^2 = -\rho t^2. \tag{2.29}$$

For the P^2 constraint, the usual prescription based on preserving diffeomorphism covariance yields $-\nabla^2 + \alpha R$ since R is the only diffeomorphism invariant scalar quadratic in derivatives [18]. Fortunately, we have a Ricci-flat ambient space, so this ambiguity is irrelevant. From the Christoffel symbols,

$$\Gamma^t_{ij} = t\left(-\hat{P}_{ij} + \rho\left(\hat{P}_{ik}\hat{P}^k_j - \frac{1}{n-4}\hat{B}_{ij}\right) + O(\rho^2)\right) \tag{2.30}$$

$$\Gamma^\rho_{t\rho} = \frac{1}{t} \tag{2.31}$$

$$\Gamma^\rho_{ij} = \hat{g}_{ij} \tag{2.32}$$

$$\Gamma^i_{tj} = \frac{1}{t}\delta^i_j \tag{2.33}$$

$$\Gamma^i_{\rho j} = -\hat{P}^i_j - \rho\left(\hat{P}_{ik}\hat{P}^k_j - \frac{1}{n-4}\hat{B}_{ij}\right) + O(\rho^2) \tag{2.34}$$

$$\Gamma^i_{jk} = \hat{\Gamma}_{ij} + 2\rho\left(\hat{P}^i_l\hat{\Gamma}^l_{ij} - \frac{1}{2}\hat{P}^{il}\left(\hat{P}_{lj,k} + \hat{P}_{lk,j} - \hat{P}_{jk,ls}\right)\right), \tag{2.35}$$

where $\hat{P}_{ij}$ and $\hat{B}_{ij}$ are the lower dimensional Schouten and Bach tensors respectively, one can calculate the scalar Laplacian

$$\frac{1}{2}P^2 \longrightarrow -\frac{1}{2}\nabla^2 = -\frac{1}{2t^2}\left(-2t\partial_t\partial_\rho + (2-n)\partial_\rho + 2\rho\partial_\rho^2 - \frac{1}{2(n-1)}R(2\rho\partial_\rho - t\partial_t) + \hat{\nabla}^2\right). \tag{2.36}$$

The scaling constraint $(X\cdot P + P\cdot X)$ is properly quantised as the commutator of the two other constraints. We have

$$\frac{1}{2}(X\cdot P + P\cdot X)\phi(t,\rho,x) \longrightarrow \left[-\frac{1}{2}\nabla^2, -\rho t^2\right]\phi(t,\rho,x) \tag{2.37}$$

$$= -\frac{1}{4t^2}\left(-4t^2 - 4t^2\rho\partial_\rho - 2t^3\partial_t + (2-n)t^2 + 4\rho t^2\partial_\rho(\rho)\partial_\rho\right)\phi(t,\rho,x) \tag{2.38}$$

$$= -\frac{1}{4t^2}\left(-4t^2 + 4t^2 - 2t^3\partial_t + (2-n)t^2 - 4t^2\right)\phi(t,\rho,x) \tag{2.39}$$

$$= \frac{1}{2}\left(t\partial_t + \frac{n+2}{2}\right)\phi(t,\rho,x), \tag{2.40}$$

which agrees with the commutator of the expressions (2.29) and (2.36) in Fefferman-Graham coordinates. This operator continues to impose the right weight on the wavefunction, while the P^2 constraint still descends to the conformal Laplace equation on M when acting on wavefunctions of the form (2.18).

$$\nabla^2\left(\delta(\rho t^2)t^{\frac{2-n}{2}}\phi(x)\right)=\nabla^2\left(\delta(\rho)t^{-\frac{n+2}{2}}\phi(x)\right) \tag{2.41}$$

$$=-\frac{t^{-\frac{n+2}{2}}}{2t^2}\Bigg(\hat{\nabla}^2+(n-2)\delta'(\rho)+(2-n)\delta'(\rho)$$

$$+\frac{1}{2(n-1)}R\left(2\rho\delta'(\rho)+\frac{n+2}{2}\right)\Bigg)\phi(x) \tag{2.42}$$

$$=t^{-\frac{n+2}{2}}\delta(\rho t^2)\left(\hat{\nabla}^2\phi(x)-\frac{n-2}{4(n-1)}R\phi(x)\right) \tag{2.43}$$

In the curved case, the inner product (2.28) generalises straightforwardly.

$$\langle f,g\rangle=-i\int_\Sigma\left(n\lrcorner D^{n+1}X\right)\delta(X^2)\frac{n\cdot\left(\bar{f}\partial g-g\partial\bar{f}\right)}{n^2} \tag{2.44}$$

$$\mapsto -i\int_\Sigma\left(t^{-1}n\lrcorner\left(t\partial_t\lrcorner t^{n+1}\sqrt{g}\,dt\,d\rho\,d^nx\right)\right)\delta(\rho t^2)\frac{t^{-1}n\cdot\left(\left(t^{\frac{2-n}{2}}\bar{f}\right)\partial\left(t^{\frac{2-n}{2}}g\right)-\left(t^{\frac{2-n}{2}}g\right)\partial\left(t^{\frac{2-n}{2}}\bar{f}\right)\right)}{n^2} \tag{2.45}$$

$$=-i\int_\Sigma\left(n\lrcorner\sqrt{g}d^nx\right)\frac{n\cdot\left(\bar{f}\partial g-g\partial\bar{f}\right)}{n^2} \tag{2.46}$$

$$=-i\int_\Sigma\sqrt{h}\frac{n\cdot\left(\bar{f}\partial g-g\partial\bar{f}\right)}{n^2}, \tag{2.47}$$

which agrees with the ordinary n-dimensional inner product given earlier.

2.4 Worldline Dirac actions

2.4.1 The conformally flat case

It is well known that quantising the four dimensional action

$$S[x,p]=\int\left(\dot{x}\cdot p+i\psi\cdot\dot{\psi}+\frac{\alpha}{2}p^2\right) \tag{2.48}$$

produces a spinor. The x and p operators are quantised from Poisson brackets in the usual way

$$\{x^\mu,p_\nu\}_P=\delta^\mu_\nu\mapsto[x^\mu,p_\nu]=i\delta^\mu_\nu \tag{2.49}$$

while the fermionic constraints satisfy anticommutators

$$\{\psi^\mu,\psi^\nu\}_P=2i\eta^{\mu\nu}\mapsto\{\psi^\mu,\psi^\nu\}=-2\eta^{\mu\nu}. \tag{2.50}$$

The fermionic operators are implemented by Dirac matrices γ^μ which satisfy these anitcommutation relations. From them we can build raising and lowering operators, $\gamma^{1\pm} = \gamma^0 \pm \gamma^2$ and $\gamma^{2\pm} = \gamma^1 \pm i\gamma^3$. We choose a ground state annihilated by γ^{i-}. There are four independent components of the spinor obtained by acting with each of the γ^{i+} at most once. The $(n+2)$-dimensional Dirac matrices are built from the n-dimensional ones using the Pauli matrices, $\gamma^\mu = (iI\otimes\sigma_1, I\otimes\sigma_2, \gamma_i\otimes\sigma_3)$, and accordingly, the number of components of the spinor doubles.

A simple way to supersymmetrise our Fefferman-Graham space action is to add a fermion kinetic term and further constraints.

$$S[X,P] = \int \left(\dot{X}\cdot P + i\psi\cdot\dot{\psi} + \frac{\alpha}{2}P^2 + \frac{\beta}{2}X^2 + \frac{\gamma}{2}(X\cdot P + P\cdot X) + \delta X\cdot\psi + \epsilon P\cdot\psi\right)dt. \tag{2.51}$$

The anticommutation relation for ψ is given by $\{\psi^\mu, \psi^\nu\} = -2\eta^{\mu\nu}$, so after quantisation ψ^μ acts as a Dirac matrix Γ^μ. Accordingly, the wavefunction has a Dirac spinor index. The multiplicative constraints $X^2\Psi = 0$ and $(X\cdot\Gamma)\,\Psi = 0$ imply that

$$\Psi(X,\psi) = \delta(X^2)(X\cdot\Gamma)\phi(X,\psi), \tag{2.52}$$

since $X\cdot\Gamma$ is a nilpotent operator on the support of the delta function. Moreover, its rank is half the dimension of the representation space, so the constraint leaves half of the components of the spinor wavefunction undetermined. This is sensible, because a Dirac spinor in two fewer dimensions has half the degrees of freedom. We will later interpret this condition as one which projects our ambient spinor to a spinor on the base.

The remaining condition $\Gamma\cdot\partial\Psi = \partial\!\!\!/\Psi = 0$ is the Dirac equation on the ambient space. It restricts to a operator on the smaller spinor space described above because the Dirac operator anticommutes with $\delta(X^2)(X\cdot\Gamma)$ provided ϕ has weight $-\frac{n}{2}$ in X.

$$\left\{\delta(X^2)(X\cdot\Gamma), \Gamma\cdot\partial\right\} = -\left[\delta(X^2), \Gamma\cdot\partial\right](X\cdot\Gamma) + \delta(X^2)\{X\cdot\Gamma, \Gamma\cdot\partial\} \tag{2.53}$$

$$= 2X^2\delta'(X^2) + 2\delta(X^2)(X\cdot\partial) + \delta(X^2)(n+2) \tag{2.54}$$

$$= 2\delta(X^2)\left(X\cdot\partial + \frac{n}{2}\right) \tag{2.55}$$

Again, to define the Hilbert space we need to give a Hermitian inner product. A natural inner product on M is

$$\langle\Psi_1, \Psi_2\rangle = i\int_\Sigma \sqrt{h}\,\bar{\psi}_1 n\!\!\!/\psi_2, \tag{2.56}$$

where as before n is a unit normal to Σ. This can again be written in flat ambient coordinates

$$\langle \Psi_1, \Psi_2 \rangle = -i \int_\Sigma \left(n \lrcorner D^{n+1} X \right) \delta(X^2) \frac{\bar{\Psi}_1 (X \cdot \Gamma)(n \cdot \Gamma) \Psi_2 - \Psi_2 (X \cdot \Gamma)(n \cdot \Gamma) \bar{\Psi}_1}{n^2}. \tag{2.57}$$

The integrand is homogenous in both X and n. It is independent of the choice of lift on n because making another choice $\tilde{n} = n + X \cdot \partial$ adds a second $(X \cdot \Gamma)$ factor to the numerator, and on the support of $\delta(X^2)$, $(X \cdot \Gamma)^2 = -X^2 = 0$. The inner product is clearly invariant under $\Psi_1 \mapsto \Psi_1 + (X \cdot \Gamma)\chi$ but is also invariant under $\Psi_2 \mapsto \Psi_2 (X \cdot \Gamma)\chi$ since n is tangent to the cone.

2.4.2 The general curved case

The Dirac equation is invariant under a conformal transformation $g \mapsto \Omega^2 g$ provided the Dirac spinor transforms as $\psi \mapsto \Omega^{-\frac{n-1}{2}} \psi$. To see this, first note that the spin connection transforms as

$$\begin{aligned} \omega_{\mu\nu\rho} &= \frac{1}{2} \left(e_{\rho a} \left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu \right) - e_{\mu a} \left(\partial_\nu e^a_\rho - \partial_\rho e^a_\nu \right) + e_{\nu a} \left(\partial_\rho e^a_\mu - \partial_\mu e^a_\rho \right) \right) \\ &\mapsto \Omega \left(\omega_{\mu\nu\rho} + g_{\mu[\nu} \partial_{\rho]} \log \Omega \right). \end{aligned} \tag{2.58}$$

So the Dirac operator transforms according to

$$\begin{aligned} \not{D}\psi &= \gamma^a e^\mu_a \left(\partial_\mu \psi + \frac{1}{4} \gamma_{[a} \gamma_{b]} \omega^{ab}_\mu \psi \right) \\ &\mapsto \Omega^{-\frac{n+1}{2}} \gamma^a e^\mu_a \left(\partial_\mu \psi + \frac{1}{4} \gamma_{[a} \gamma_{b]} \omega^{ab}_\mu \psi - \frac{n-1}{2} \partial_\mu (\log \Omega) \psi + \frac{1}{2} \gamma_{[b} \gamma_{c]} e^b_\mu e^{c\nu} \partial_\nu (\log \Omega) \psi \right) \\ &\mapsto \Omega^{-\frac{n+1}{2}} \not{D}\psi. \end{aligned} \tag{2.59}$$

Although the Laplacian is the square of the Dirac operator, it does not follow from this (and is indeed untrue) that the plain Laplacian is conformally invariant because $\not{D}\psi$ is a spinor of conformal weight $-\frac{n+1}{2}$ rather than $-\frac{n-1}{2}$.

Writing the $(n+2)$-dimensional gamma matrices $\Gamma_A = (\Gamma_1, \Gamma_2, \Gamma_a) = (iI \otimes \sigma_1, I \otimes \sigma_2, \gamma_i \otimes \sigma_3)$ in terms of the n-dimensional gamma matrices γ^i on the base manifold and the Pauli matrices σ_i, we find that non-vanishing parts of the Fefferman-Graham spin connection are

$$\omega_i^{a1} = e_i^a - \frac{1}{2} P_i^j e_j^a, \tag{2.60}$$

$$\omega_i^{a2} = e_i^a + \frac{1}{2} P_i^j e_j^a \tag{2.61}$$

and all other components vanish. As with the bosonic constraints, X is interpreted as the Euler vector field generating scaling transformations. The fermionic constraints generalise to the curved case as follows.

$$X \cdot \psi \longrightarrow X^\mu e_\mu^A \Gamma_A = t\left(\rho I \otimes i\sigma_- + \frac{1}{2} I \otimes i\sigma_+\right), \tag{2.62}$$

$$P \cdot \psi \longrightarrow \nabla = \frac{1}{t}\left(\hat{\nabla} \otimes \sigma_3 - I \otimes i\sigma_- t\partial_t + I \otimes \left(\rho i\sigma_- - \frac{1}{2} i\sigma_+\right)\partial_\rho - \frac{n}{2} I \otimes i\sigma_- - \frac{R}{8(n-1)} iI \otimes \sigma_+\right). \tag{2.63}$$

with the vierbeins

$$e_t^A = \begin{pmatrix} \rho + \frac{1}{2} \\ \rho - \frac{1}{2} \\ 0 \end{pmatrix}, \; e_\rho^A = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \; e_i^A = \begin{pmatrix} 0 \\ 0 \\ t(\hat{e}_i^a - \rho P_i^j \hat{e}_j^a + \cdots) \end{pmatrix}. \tag{2.64}$$

It can be checked directly that the operators $X \cdot \Psi$ and $P \cdot \Psi$ obey the same (anti) commutation relations as in the conformally flat case on the cone $\rho = 0$. In particular, ordering ambiguities in $P \cdot \Psi$ arising from the position-dependent vierbein are resolved by requiring that $(P \cdot \Psi)^2 = P^2$, which has already been quantised unambiguously as the Laplacian. This is another place in which ordering ambiguities might have arisen on the base manifold, but the descent of our theory from the ambient space ensures conformally invariance without the imposition of artificial choices.

Our wavefunctions will take the form the form (2.52), which are represented in Fefferman-Graham coordinates as

$$\Psi(t, \rho, x) = t^{-\frac{n+2}{2}} \delta(\rho) I \otimes i\sigma_+ \psi(x), \tag{2.65}$$

where here we have the ρ terms vanish on the support of $\delta(\rho t^2) = t^{-2}\delta(\rho)$. On such wavefunctions, the operators $X \cdot \Psi$ and $P \cdot \Psi$ take the simplified forms

$$X \cdot \psi \longrightarrow \frac{1}{2} t I \otimes i\sigma_+, \tag{2.66}$$

$$P \cdot \psi \longrightarrow \frac{1}{t} \hat{\nabla} \otimes \sigma_3, \tag{2.67}$$

using $\rho\delta'(\rho) = -\delta(\rho)$, with all terms in ρ vanishing and the terms proportional to $I \otimes \sigma_+$ acting to give zero on the wavefunction.

The descended equation is therefore

$$\delta(\rho)\hat{\nabla} \otimes i\sigma_+ \psi(x) = 0. \tag{2.68}$$

Again, we interpret the σ_+ factor as a projection onto the subspace of spinors which originate on the base manifold. This is the Dirac equation on the base manifold.

2.5 Discussion

In this section, we have constructed worldline actions for the free massless scalar and Dirac fermion on the Fefferman-Graham embedding space with constraints which allow them to descend to theories on the base manifold. Since the Fefferman-Graham space is conformally invariant, the descended theories are conformal. From the equations of motion of these two theories, we obtain perhaps the two most famous and most important conformal equations - the conformal wave equation and the Dirac equation. It seems likely that other conformal theories, and their accompanying equations of motion, could be obtained from the descent of other constrained theories defined on the Fefferman-Graham space.

2.5.1 Obstructions

The ambient metric is constructed order-by-order in ρ so that the result is Ricci-flat. It is shown in [51] that it is possible to all orders for odd n, and up to order $\frac{n-2}{2}$ for even n. The existence and uniqueness of a solution beyond this depends on the vanishing of a particular *obstruction tensor*. Naturally, this tensor is a conformal invariant, but explicit formulae are difficult to obtain for all n. In the case $n = 4$, it is known to be the Bach tensor, a classical conformal invariant given by

$$B_{ij} = P_{kl} W_{i\ j}^{\ k\ l} + \nabla^k \nabla_i P_{jk} - \nabla^2 P_{ij}. \tag{2.69}$$

The Bach tensor is the metric variation of the 4 dimensional conformal anomaly

$$\int_M W_{abcd} W^{abcd} \sqrt{g} d^4 x \tag{2.70}$$

The construction of worldline actions depended on the vanishing of this obstruction, but the conformal Laplace equation exists for any metric. This is because it uses the Fefferman-Graham expansion only to low order.

It would be interesting to investigate connections between the obstruction tensor and conformal anomalies.

Chapter 3

Twistor methods for AdS_5

3.1 Introduction

In recent years, twistors have played an important role in studying scattering amplitudes of four-dimensional gauge and gravitational theories. The fundamental tool underlying these investigations is the (linear) Penrose transform [84, 48]. This asserts that solutions to massless, free field equations on four-dimensional Minkowski space-time may be described in terms of essentially arbitrary holomorphic functions on twistor space, with the homogeneity of the function determining the helicity of the space-time field. The asymptotic states in scattering processes are taken to obey such free field equations, so twistors are a natural language in which to construct amplitudes.

Twistors also provide a natural arena in which to study four-dimensional CFTs. This is because twistor space carries a natural action of $SL(4,\mathbb{C})$, the (four-fold cover of the) complexification of the space-time conformal group. Here, twistors are closely related to the 'embedding space' formalism used in *e.g.* [43, 111, 82, 53, 81, 92, 37] and are particularly useful when considering operators with non-integer spin [55, 52]. In the context of $\mathcal{N} = 4$ SYM, twistor methods have been applied to correlation functions of local gauge invariant operators in *e.g.* [4, 1, 33, 70, 34].

By the AdS/CFT correspondence, many four-dimensional CFTs have a dual description as a theory of gravity in five-dimensional anti-de Sitter space [73, 57, 114]. Given the utility of twistor theory on the boundary side of this correspondence, is natural to ask if it can also be applied in the bulk.

In this chapter, we begin an investigation of the role of twistors in AdS_5, following earlier mathematical work in [14]. After briefly reviewing various descriptions of AdS and its complexification, in section 3.2 we describe its twistor space and the corresponding incidence relations. Remarkably, the *twistor* space of AdS$_5$ turns out to be the same as the *ambitwistor* space of the boundary space-time. We explore and elucidate this construction in detail. In section 3.3 we consider the Penrose transform for free fields on AdS$_5$. Unlike in flat space-time, we show that it is straightforward to describe fields with non-zero mass as well as non-zero spin. From the point of view of AdS/CFT, the most important free fields are bulk-to-boundary propagators and we provide explicit twistor descriptions of these in section 3.4, concentrating on spin-0 and spin-$\frac{1}{2}$. We also construct simple twistor actions for these fields and verify that, when evaluated on bulk-to-boundary propagators, they reproduce the expected form for 2-point correlation functions of boundary operators of the expected conformal weights and spins. We hope that these results will provide a useful starting-point for a twistor reformulation of Witten diagrams.

This chapter is based on collaborative work and primarily reproduced from [8].

3.2 Geometry

The geometry of anti-de Sitter space (or hyperbolic space) is an old and well-studied topic. For the purposes of describing twistor theory in the context of five-dimensional AdS, a particular description of hyperbolic geometry in terms of an open subset of projective space will prove useful. While this description is standard, it is not often utilized in the physics literature so we being with a brief review of AdS$_5$ geometry from a projective point of view. The twistor space of AdS$_5$ and various aspects of its geometry are then discussed.

3.2.1 AdS$_5$ geometry from projective space

Consider the five-dimensional complex projective space $\mathbb{CP}^5$, charted by homogeneous coordinates encoded in a skew symmetric 4×4 matrix $X^{AB} = X^{[AB]}$ with the identification $X \sim \lambda X$ for any $\lambda \in \mathbb{C}^*$. For a (holomorphic) metric written in terms of these homogeneous coordinates to be well-defined on $\mathbb{CP}^5$ it must be invariant with respect to the scaling $X \to \lambda X$ and have no components along this scaling direction (*i.e.*, the metric must not 'point off' $\mathbb{CP}^5$ into $\mathbb{C}^6$). The simplest metric satisfying these conditions is

$$\mathrm{d}s^2 = -\frac{\mathrm{d}X^2}{X^2} + \left(\frac{X\cdot \mathrm{d}X}{X^2}\right)^2 , \tag{3.1}$$

where skew pairs of indices are contracted with the Levi-Civita symbol, ϵ_{ABCD}. This line element is obviously scale invariant, and furthermore has no components in the scale direction. The latter fact follows since the contraction of (3.1) with the Euler vector field $\Upsilon = X \cdot \frac{\partial}{\partial X}$ vanishes.

Although this metric is projective (in the sense that it lives on $\mathbb{CP}^5$ rather than $\mathbb{C}^6$), it is not global: (3.1) becomes singular on the quadric

$$M = \left\{X \in \mathbb{CP}^5 | X^2 = 0\right\} \subset \mathbb{CP}^5 .$$

So (3.1) gives a well-defined metric on the open subset $\mathbb{CP}^5 \setminus M$. It is a fact that $\mathbb{CP}^5 \setminus M$ equipped with this metric is equivalent to complexified AdS_5, with the quadric M corresponding to the four-dimensional conformal boundary. Real AdS_5, along with a choice of signature (Lorentzian or Euclidean, for instance) is specified by restricting the metric to a particular real slice of $\mathbb{CP}^5$ – or equivalently, imposing some reality conditions on X^{AB}. We will be explicit about these reality conditions below.

To see that (3.1) really describes AdS_5, it suffices to show that it is equivalent to other well-known models of hyperbolic geometry. It is straightforward to see that the metric can be rewritten as

$$\mathrm{d}s^2 = -\epsilon_{ABCD}\, \mathrm{d}\left(\frac{X^{AB}}{|X|}\right) \mathrm{d}\left(\frac{X^{CD}}{|X|}\right) = -\epsilon_{ABCD}\, \mathrm{d}\mathscr{X}^{AB} \mathrm{d}\mathscr{X}^{CD} , \tag{3.2}$$

where $\mathscr{X}^{AB} := X^{AB}/|X|$ with $|X| := \sqrt{X^2}$. The coordinates $\mathscr{X}^{AB}$ are invariant under scalings of X^{AB}, so they give coordinates on $\mathbb{C}^6$ obeying $\mathscr{X}^2 = 1$. Since (3.2) is just the flat metric on $\mathbb{C}^6$, the original metric on $\mathbb{CP}^5 \setminus M$ describes a geometry equivalent to the quadric $\mathscr{X}^2 = 1$ in $\mathbb{C}^6$. With an appropriate choice of reality conditions, this is the well-known model of AdS_5 as the hyperboloid in $\mathbb{R}^6$.

To obtain the conformal compactification of AdS_5, one includes a conformal boundary isometric to the one-point compactification of 4-dimensional complexified flat space; with appropriate reality conditions this is topologically S^4. We wish to identify this boundary with the quadric $M \subset \mathbb{CP}^5$ on which (3.1) becomes singular. A point $X \in M$ satisfies $X^2 = 0$ and hence $\det X = 0$. Since X^{AB} is antisymmetric, non-zero and degenerate, it must have rank 2 and so can be written as the skew of two 4-vectors,

$$X^{AB} = C^{[A} D^{B]} . \tag{3.3}$$

However, X is projectively invariant under the (separate) transformations

$$(C,D) \mapsto (C, D+\alpha C) , \quad (C,D) \mapsto (C+\beta D, D) , \quad (C,D) \mapsto (\gamma C, D) , \quad (C,D) \mapsto (C, \delta D)$$

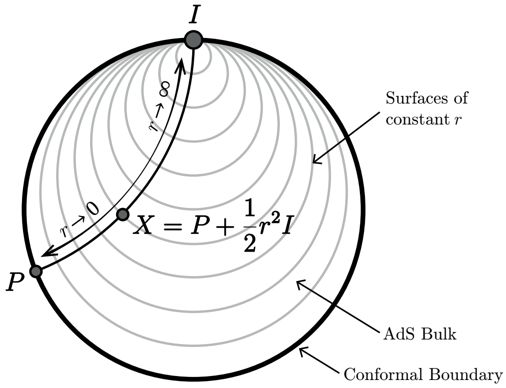

Fig. 3.1 Parametrization of AdS space by Poincaré coordinates. The coordinate r controls the distance to the conformal boundary.

for $\alpha, \beta \in \mathbb{C}$, $\gamma, \delta \in \mathbb{C}^*$. Performing a sequence of these transformations allows us to assume that C and D take the form

$$C = \begin{pmatrix} a \\ c \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} b \\ d \\ 0 \\ 1 \end{pmatrix}, \tag{3.4}$$

where some of a, b, c and d may be infinite, and after which there is no remaining freedom. Thus, the general form of a boundary point is

$$X^{AB}_{\text{bdry}} = \begin{pmatrix} \frac{1}{2} x^2 \epsilon^{\dot\alpha\dot\beta} & x^{\dot\alpha}{}_{\beta} \\ -x_{\alpha}{}^{\dot\beta} & \epsilon_{\alpha\beta} \end{pmatrix}, \tag{3.5}$$

where $\alpha, \dot\alpha, \ldots$ are dotted and un-dotted two component $\mathrm{SL}(2, \mathbb{C})$ spinors. The four components of $x^{\alpha\dot\alpha}$ encode the four degrees of freedom in (3.4). Including the point 'at infinity,' represented by the *infinity twistor*

$$I^{AB} = \begin{pmatrix} \epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.6}$$

gives the one-point compactification of four-dimensional flat space, with $x^{\alpha\dot\beta}$ serving as the usual spinor helicity coordinates. Thus, $M = \{X^2 = 0\}$ is identified with the S^4 conformal boundary of AdS$_5$. The relationship between four-dimensional space-time and simple points in $\mathbb{CP}^5$ is well-established, having appeared in various places in a variety of different guises (*e.g.*, [43, 64, 111]).

It is straightforward to obtain other well-known models of AdS_5 from the projective one. For example, the Klein model of hyperbolic space is obtained by simply writing the metric (3.1) using inhomogeneous coordinates on a patch where one of the X^{AB} is non-vanishing. One of the models of AdS used most widely in physical applications is the Poincaré model; in Euclidean signature these are global coordinates, and the metric takes the form:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + \mathrm{d}x_{\alpha\dot{\alpha}}\mathrm{d}x^{\alpha\dot{\alpha}}}{r^2}\,, \tag{3.7}$$

with the conformal boundary corresponding to the region where $r \to 0$.

To obtain Poincaré coordinates from the projective model, it suffices to choose a parametrization for X^{AB} in terms of a variable boundary point, P^{AB}, of the form (3.5) and some fixed boundary point. It is convenient to let this fixed boundary point be precisely the infinity twistor (3.6), and write:

$$X^{AB} = P^{AB} + \frac{r^2}{2} I^{AB}\,. \tag{3.8}$$

As $r \to 0$, we approach a boundary point P, but as $r \to \infty$ with P constant, we approach the fixed infinity twistor. Surfaces of constant $r > 0$ correspond to spheres in the bulk of AdS_5 which touch the boundary only at I. As $r \to 0$ this sphere approaches the whole boundary, but as $r \to \infty$ it shrinks to the single point I. Note that $X^2 = r^2$, so r controls the distance from the conformal boundary. Plugging the parametrization (3.8) into (3.1) leads directly to the Poincaré metric (3.7) after a rescaling of the boundary coordinates $x^{\alpha\dot{\alpha}}$ by an overall factor of two. This Poincaré parametrization will prove useful later when we want to check that certain expressions derived from twistor methods correspond to well-known formulae on space-time.

Let us conclude our review of AdS_5 geometry with a brief discussion of the reality conditions which can be imposed on the X^{AB} to obtain a real space-time with explicit signature. This is best understood by viewing the metric in terms of the scale-free $\mathscr{X}^{AB}$, constrained to be $\mathscr{X}^2 = 1$, as in (3.2). On $\mathbb{C}^6$, there are two representations of chiral spinors with four components; these are dual to each other, and the bundles of such spinors are denoted by $\mathbb{S}^A$, $\mathbb{S}_A$ respectively. The coordinates $\mathscr{X}^{AB}$ live in the anti-symmetric square of the first of these: $\mathbb{S}^A \wedge \mathbb{S}^B$.

Reality conditions on the $\mathscr{X}^{AB}$ – and hence the homogeneous coordinates X^{AB} – correspond to a reality structure on these spinor bundles [75, 90]. Introduce a quaternionic conjugation acting on $Z^A \in \mathbb{S}^A$ by

$$Z^A = (\mu^{\dot{0}}, \mu^{\dot{1}}, \lambda_0, \lambda_1) \mapsto \hat{Z}^A = (-\bar{\mu}^{\dot{1}}, \bar{\mu}^{\dot{0}}, -\bar{\lambda}_1, \bar{\lambda}_0)\,,$$

which squares to minus the identity: $\hat{\hat{Z}}^A = -Z^A$. Clearly, there are no real spinors under the $\hat{\cdot}$-operation, but this conjugation does act involutively on $\mathscr{X}^{AB}$. Restricting to the real slice $\hat{\mathscr{X}}^{AB} = \mathscr{X}^{AB}$ inside $\mathbb{C}^6$ turns (3.2) into the flat metric on $\mathbb{R}^{1,5}$. This, along with the condition that $\mathscr{X}^2 = 1$ indicates that these reality conditions describe *Euclidean* AdS$_5$ (the hyperbolic space $\mathbb{H}_5$).

To obtain Lorentzian AdS$_5$, a different reality condition is required. Instead of the quaternionic conjugation, one can take ordinary complex conjugation which exchanges the spinor representations:

$$Z^A \mapsto \overline{Z^A} = \bar{Z}_A .$$

The reality condition on $\mathscr{X}^{AB}$ is then

$$\overline{\mathscr{X}^{AB}} = \bar{\mathscr{X}}_{AB} = \frac{1}{2}\epsilon_{ABCD}\mathscr{X}^{CD} .$$

This real slice results in the flat metric on $\mathbb{R}^{2,4}$, and thus Lorentzian AdS$_5$ as the hyperboloid.

3.2.2 The twistor space of AdS$_5$

It is an interesting fact that the twistor space of AdS$_5$ is the same geometric space as the projective *ambitwistor* space of the complexified, four-dimensional conformal boundary. In any number of dimensions, the projective ambitwistor space of a Riemannian manifold $M_{\mathbb{R}}$ is the space of complex null geodesics in the complexified manifold M [112, 66, 71, 113]. In the case that $M_{\mathbb{R}} = S^4$, this ambitwistor space can be written as a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$:

$$Q = \left\{(Z^A, W_B) \in \mathbb{CP}^3 \times (\mathbb{CP}^3)^* \,|\, Z\cdot W = 0\right\} , \tag{3.9}$$

where Z^A, W_B are homogeneous coordinates on the two (dual) copies of $\mathbb{CP}^3$, each with its own scaling freedom. The ambitwistor correspondence relates a point in M to a $\mathbb{CP}^1 \times (\mathbb{CP}^1)^* \subset Q$, which can be thought of as the complexified sphere of null directions through that point.

The quadric Q also serves as the *twistor* space of (complexified) AdS$_5$.[1] The usual twistor correspondence relates a space-time point to an extended geometric object in twistor space, with the intersection theory of these objects encoding the conformal structure of the space-time. To formulate this correspondence, we relate AdS$_5$ to Q by the

[1]This fact has been known for some time; a mathematical treatment was given by [14], and some aspects have also appeared in the physics literature [88, 93, 11].

incidence relations:

$$Z^A = X^{AB} W_B \,, \tag{3.10}$$

where X^{AB} describes a point in AdS$_5$. It is easy to see that for a fixed (up to scale) X, (3.10) defines a $\mathbb{CP}^3_X \subset \mathbb{CP}^3 \times (\mathbb{CP}^3)^*$; the fact that $\mathbb{CP}^3_X \subset Q$ follows from the anti-symmetry of X^{AB} (*i.e.*, the incidence relation preserves $Z \cdot W = 0$). Further, since $X^{AB} \in \mathbb{CP}^5 \setminus M$ it has no kernel so the incidence relation is non-degenerate.

For Q equipped with (3.10) to be the correct twistor space, the geometry of the incidence relations should capture the conformal geometry of AdS$_5$. To see this, consider two distinct points $X, Y \in \mathbb{CP}^5 \setminus M$ and the corresponding $\mathbb{CP}^3_X, \mathbb{CP}^3_Y \subset Q$. Generically, $\mathbb{CP}^3_X$ and $\mathbb{CP}^3_Y$ will intersect in two projective lines in Q. To see this, note that $\mathbb{CP}^3_X \cap \mathbb{CP}^3_Y$ consists of the points $(Z, W) \in \mathbb{CP}^3_X$ for which $(X - tY)^{AB} W_B = 0$ for some $t \in \mathbb{C}^*$. The antisymmetric matrix $(X - tY)^{AB}$ has a non-trivial kernel whenever it squares to zero, in which case its kernel is of complex projective dimension one. This shows that $\mathbb{CP}^3_X \cap \mathbb{CP}^3_Y$ consists of some number of copies of $\mathbb{CP}^1$. To establish how many, it is useful to write the intersection condition in a scale-free way:

$$\left(\frac{X^{AB}}{|X|} - s\frac{Y^{AB}}{|Y|}\right) W_B = 0 \quad \Leftrightarrow \quad \left(\frac{X}{|X|} - s\frac{Y}{|Y|}\right)^2 = 0 \,.$$

This give a quadratic equation in s which has two distinct solutions given by

$$s_{\pm} = \frac{X \cdot Y}{|X||Y|} \pm \sqrt{\left(\frac{X \cdot Y}{|X||Y|}\right)^2 - 1} \,, \tag{3.11}$$

each of which corresponds to an intersection of $\mathbb{CP}^3_X \cap \mathbb{CP}^3_Y$ isomorphic to $\mathbb{CP}^1$.

Generically, these two lines do not themselves intersect because Y^{AB} is non-degenerate. However, when

$$\frac{X}{|X|} \cdot \frac{Y}{|Y|} = 1, \tag{3.12}$$

these two solutions degenerate into a single $\mathbb{CP}^1$. Since the geodesic distance $d(X, Y)$ between two points in AdS$_5$ satisfies

$$\cosh\left(d(X, Y)\right) = \frac{X}{|X|} \cdot \frac{Y}{|Y|}, \tag{3.13}$$

the pairs of points satisfying (3.12) are precisely those which are null separated. In other words, two points in $\mathbb{CP}^5 \setminus M$ are null separated in the AdS conformal structure if and only if their corresponding $\mathbb{CP}^3$s intersect in a single line in twistor space.

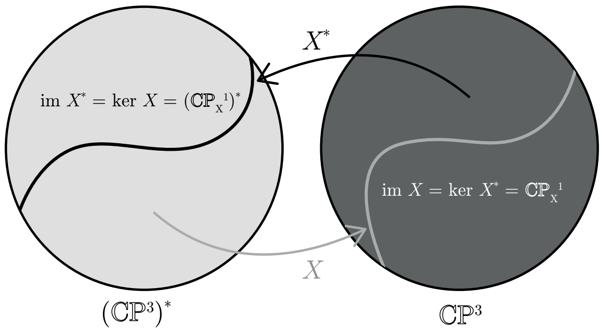

Fig. 3.2 Relationship between the linear map X corresponding to a boundary point and its dual. Both maps determine the canonical $\mathbb{CP}^1 \times \left(\mathbb{CP}^1\right)^*$ inside Q.

A null structure on a (complexified) Lorentzian manifold determines the metric up to a conformal factor. The null structure given by this degeneracy condition is a canonical choice, so we recover the AdS_5 metric (3.1) up to the conformal factor. This factor is fixed by making the canonical choice of holomorphic volume form on the $\mathbb{CP}^3_X$ corresponding to a space-time point:

$$\mathrm{D}^3 W := \epsilon^{ABCD} W_A \mathrm{d}W_B \wedge \mathrm{d}W_C \wedge \mathrm{d}W_D \,, \tag{3.14}$$

which sets the overall conformal factor in (3.1) to unity.

What happens in twistor space if X^{AB} corresponds to a point on the conformal boundary? This means that $X^2 = 0$ so X^{AB} has a non-trivial kernel and the incidence relations (3.10) become degenerate. In particular, since Z^A are homogeneous coordinates on $\mathbb{CP}^3$, they cannot all be simultaneously zero – but there are now solutions of $X^{AB} W_B = 0$. The space of such solutions has complex projective dimension one, as does the image of X^{AB}_{bdry} when viewed as a linear map on $(\mathbb{CP}^3)^*$. So for a boundary point X_{bdry} the degenerate incidence relations are replaced by the linear map

$$X_{\mathrm{bdry}} : \left(\mathbb{CP}^3\right)^* \setminus (\mathbb{CP}^1_X)^* \to \mathbb{CP}^1_X \,, \tag{3.15}$$

where

$$(\mathbb{CP}^1_X)^* = \left\{X^{AB}_{\mathrm{bdry}} W_B = 0\right\} \subset (\mathbb{CP}^3)^* \,, \qquad \mathbb{CP}^1_X = \left\{X^{\mathrm{bdry}}_{AB} Z^B = 0\right\} \subset \mathbb{CP}^3 \,,$$

are the kernel and image of the linear map, respectively.

In fact, boundary points X_{bdry} are in one-to-one correspondence with sets $\mathbb{CP}^1_X \times (\mathbb{CP}^1_X)^* \subset Q$. The choice of $\mathbb{CP}^1_X \times (\mathbb{CP}^1_X)^*$ determines both the kernel and image of X^{AB}_{bdry}, and any antisymmetric 4×4 matrix is fixed by these up to an overall scale. This scale is irrelevant because X^{AB} describes a point in the projective space $\mathbb{CP}^5$. More generally,

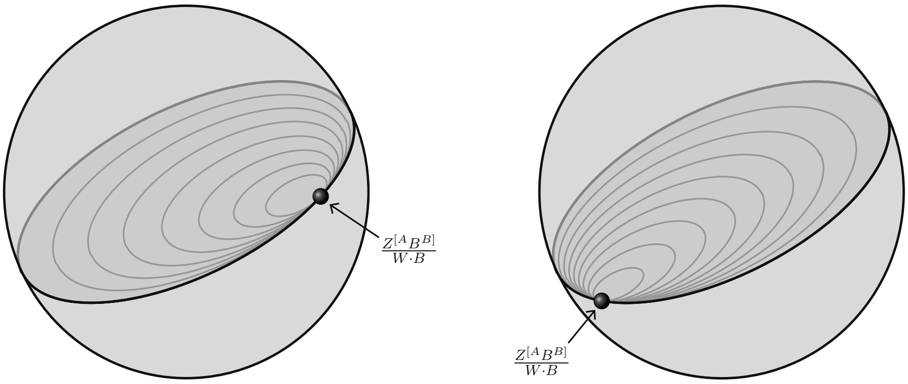

Fig. 3.3 The totally null set of points in spacetime $\mathbb{CP}^5$ corresponding to a twistor point (Z, W) for two different choices of B. We view B as fixed and vary A, tracing out a three-dimensional space of solutions. Changing B alters the parametrization of this solution space, but not the set itself.

any subset $\mathbb{CP}^1_X \times (\mathbb{CP}^1_Y)^* \subset \mathbb{CP}^3 \times (\mathbb{CP}^3)^*$ can be specified by two points on the conformal boundary $x^{\alpha\dot{\alpha}}$, $y^{\alpha\dot{\alpha}}$ as in (3.5). The condition that this subset lies inside Q, namely that $Z\cdot W = 0$ imposes the constraint $x^{\alpha\dot{\alpha}} = y^{\alpha\dot{\alpha}}$. Hence, the two lines $\mathbb{CP}^1_X \times (\mathbb{CP}^1_Y)^* \subset Q$ correspond to the *same* point on the four-dimensional boundary.

This establishes the geometry of twistor space for both the bulk and boundary of AdS_5. A point in the bulk corresponds to a $\mathbb{CP}^3$ inside Q; for boundary points this correspondence degenerates to give the standard ambitwistor relation between a point on the boundary and a $\mathbb{CP}^1 \times (\mathbb{CP}^1)^*$ inside Q.

It is equally natural to ask for the twistor correspondence in the other direction: what does a point in twistor space correspond to in space-time? Given fixed $(Z, W) \in Q$, we want to know which space-time points X satisfy the incidence relations

$$Z^A = X^{AB} W_B\,.$$

The solution set consists of points of the form

$$X^{AB} = \frac{Z^{[A}B^{B]}}{W\cdot B} + \epsilon^{ABCD} A_C W_D, \tag{3.16}$$

where A_C is an arbitrary parameter and B^B is an arbitrary twistor with $B\cdot W \neq 0$. Transformations of the form $A \mapsto A + \alpha W$ leave X invariant so the space of solutions is three-dimensional. Making a different choice of B can be accommodated by a redefinition of A, so B contributes no further degrees of freedom. Moreover, any tangent vector to this set is

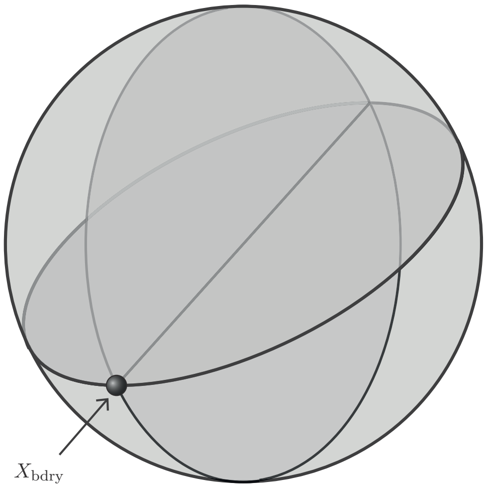

Fig. 3.4 The set of points in spacetime $\mathbb{CP}^5$ corresponding to two different twistor points (Z,W) and $(\tilde{Z},\tilde{W})$. The intersection is generically one-dimensional, but if $Z\cdot\tilde{W}=0$ and $\tilde{Z}\cdot W=0$, then it is two dimensional and the closure includes the boundary point whose canonical $\mathbb{CP}^1_X\times\left(\mathbb{CP}^1_X\right)^*\subset Q$ contains (Z,W) and $(\tilde{Z},\tilde{W})$.

a null vector of the form $\epsilon^{ABCD}(\delta A)_C W_D$, where δA is a displacement in the parameter A. Thus, a point in twistor space corresponds to a three-plane in $\mathbb{CP}^5$ and hence AdS_5.

How can two such three-planes intersect? Let $(Z,W),(\tilde{Z},\tilde{W})\in Q$ be distinct twistor points; the general projective solution to the simultaneous equations

$$\begin{aligned} Z^A &= X^{AB}W_B, \\ \widetilde{Z}^A &= X^{AB}\widetilde{W}_B \end{aligned} \tag{3.17}$$

is

$$\begin{aligned} X^{AB} = \alpha&\left(Z^{[A}B^{B]}\frac{\widetilde{Z}\cdot W}{W\cdot B}-\widetilde{Z}^{[A}\widetilde{B}^{B]}\frac{Z\cdot\widetilde{W}}{\widetilde{W}\cdot\widetilde{B}}+B^{[A}\widetilde{B}^{B]}\frac{(Z\cdot\widetilde{W})(\widetilde{Z}\cdot W)}{(B\cdot W)(\widetilde{B}\cdot\widetilde{W})}\right) \\ &+\gamma\epsilon^{ABCD}W_C\widetilde{W}_D, \end{aligned} \tag{3.18}$$

where $\alpha\neq 0$ and $B,\tilde{B}$ are such that $B\cdot\tilde{W}=0$ and $\tilde{B}\cdot W=0$ while $B\cdot W\neq 0$, $\tilde{B}\cdot\tilde{W}\neq 0$. This solution is parametrized by the two complex numbers α,γ, or equivalently, a projective line. So for all pairs (Z,W) and $(\tilde{Z},\tilde{W})$ the corresponding null three-planes intersect in a line in $\mathbb{CP}^5$.

However, if $Z\cdot\tilde{W}=0$ and $\tilde{Z}\cdot W=0$, then further solutions are possible. In this case, the general solution is

$$X^{AB}=\alpha\frac{Z^{[A}B^{B]}}{W\cdot B}+\beta\frac{\widetilde{Z}^{[A}\widetilde{B}^{B]}}{\widetilde{W}\cdot\widetilde{B}}+\gamma\epsilon^{ABCD}W_C\widetilde{W}_D, \tag{3.19}$$

with $\alpha,\beta\neq 0$. This gives a two-dimensional projective space of solutions, parametrized by homogeneous coordinates (α,β,γ). Note that the conditions $Z\cdot\tilde{W}=0$ and $\tilde{Z}\cdot W=0$ mean that (Z,W) and $(\tilde{Z},\tilde{W})$ lie inside $\mathbb{CP}^1_X\times\left(\mathbb{CP}^1_X\right)^*\subset Q$ for a boundary point

$$X^{AB}_{\text{bdry}}=\epsilon^{ABCD}W_C\widetilde{W}_D\,.$$

This point is in the closure of the two-dimensional intersection of their three-planes (3.19) but not in the solution space itself, since it requires $(\alpha,\beta,\gamma)=(0,0,1)$.[2]

3.3 The Penrose transform

A basic property of twistor theory in any number of dimensions is its ability to encode fields living on space-time in terms of geometric data on twistor space. In four space-time dimensions the basic tool in this regard is the *Penrose transform*, relating solutions of the zero-rest-mass equations to certain cohomology classes on twistor space [84, 48]. It is less widely known that the Penrose transform extends to any number of space-time dimensions, where cohomology of the corresponding twistor space encodes solutions to certain equations on space-time [17].

We want to describe fields on AdS_5 in terms of some geometric data on the twistor space Q. Simple examples of such fields are massive scalars or spinors, which obey field equations

$$\Box_{\text{AdS}}\Phi-m^2\Phi=0\,,\qquad \not{D}_{\text{AdS}}\Psi=m\Psi\,, \tag{3.20}$$

respectively, with $\Box_{\text{AdS}}$ the AdS_5 Laplacian and $\not{D}_{\text{AdS}}$ the AdS_5 Dirac operator. For such scalar and spinor fields in AdS_5 it is well-known that their masses obey relations:

$$m^2=\Delta(\Delta-4)\,, \tag{3.21}$$

for the scalar, and

$$|m|=\Delta-2\,, \tag{3.22}$$

[2]It is interesting to contrast this against the situation for the twistor space of $\mathbb{C}^6$. There twistor points define totally null 3-planes which do not intersect generically, and only intersect in a line if their twistor points obey a nullity relation akin to $Z\cdot\tilde{W}+\tilde{Z}\cdot W=0$ [65].

for the spinor. The parameter Δ controls the asymptotic behaviour of the fields near the AdS boundary, and is also the conformal dimension of the local operator in the boundary CFT_4 [114].

3.3.1 Scalars: Direct and indirect transform

Functions of specific homogeneity in Z and W form a natural set of line bundles on Q. In particular, denote the line bundle of holomorphic functions scaling as

$$f(\alpha Z, \beta W) = \alpha^m \beta^n f(Z, W), \qquad \alpha, \beta \in \mathbb{C}^*,$$

by $\mathcal{O}(m,n) \to Q$. The line bundles $\mathcal{O}(m,n)$ can be tensored with other bundles over Q to form weighted bundles of geometric objects with the specified scaling properties.

For some fixed scaling dimension Δ, consider a $(0,3)$-form on Q taking values in $\mathcal{O}(-\Delta, \Delta-4)$, denoted by $f \in \Omega^{0,3}(Q, \mathcal{O}(-\Delta, \Delta-4))$. The bundle $\mathcal{O}(-\Delta, \Delta-4)$ is only well-defined if $\Delta \in \mathbb{Z}$, but this is consistent with the expected conformal dimensions of boundary operators dual to bulk scalars. Restricting f to the $\mathbb{CP}^3_X \subset Q$ corresponding to the AdS_5 point X is accomplished simply by imposing the incidence relations:

$$f(Z^A, W_B)|_X = f(X^{AC} W_C, W_B).$$

So $f|_X$ is a $(0,3)$-form on $\mathbb{CP}^5 \times \mathbb{CP}^3_X$ which is homogeneous of degree $-\Delta$ in X and -4 in W. Integrating $f|_X$ over $\mathbb{CP}^3_X$, we define

$$\Phi(X) = |X|^\Delta \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge f|_X. \tag{3.23}$$

Clearly, Φ is homogeneous of degree zero in X (*i.e.*, $X \cdot \partial\Phi = 0$), and hence a well-defined scalar field on AdS_5 rather than a section of some line bundle over $\mathbb{CP}^5$. Further, it is an easy consequence of the incidence relations that Φ obeys

$$\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X} \left(|X|^{-\Delta}\Phi\right) = 0,$$

if and only if f is holomorphic, $\bar\partial f = 0$. Since any f which is $\bar\partial$-exact integrates to zero, we see that $\Phi(X)$ is determined by the cohomology class $[f] \in H^{0,3}(Q, \mathcal{O}(-\Delta, \Delta-4))$ on twistor space.

A straightforward calculation, given in the appendix, reveals that

$$X \cdot \frac{\partial}{\partial X}\Phi = 0 = \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\left(|X|^{-\Delta}\Phi\right) \quad \Leftrightarrow \quad \Box_{\mathrm{AdS}}\Phi = \Delta(\Delta-4)\Phi\,.$$

Thus, any f which is a cohomology class defines a solution to the scalar equation of motion with appropriate scaling dimension Δ via the integral construction (3.23). An argument in homological algebra can be used to show that in fact *every* massive scalar on AdS_5 – subject to suitable analyticity conditions – can be represented in this way [17, 14]. We refer to this correspondence as the *direct Penrose transform*:

$$H^{0,3}(Q,\mathcal{O}(-\Delta,\Delta-4)) \cong \{\Phi(X) \text{ on } \mathrm{AdS}_5 \,|\, \Box_{\mathrm{AdS}}\Phi = \Delta(\Delta-4)\Phi\}\,, \tag{3.24}$$

the isomorphism being realized from left to right by the integral formula (3.23).

Unlike in four-dimensions, the Penrose transform in $d > 4$ is not unique. For AdS_5, this non-uniqueness takes two different forms. The first of these is rather trivial, following from the fact that for bulk points X, the incidence relations can be inverted:

$$Z^A = X^{AB}W_B \Leftrightarrow W_B = \frac{X_{BC}}{X^2}Z^C\,. \tag{3.25}$$

These inverted relations associate a 'dual' $\mathbb{CP}^3$ to X which is now parametrized by Z rather than W; we denote this dual by $(\mathbb{CP}^3_X)^*$. Interchanging homogeneities of Z and W for a cohomology class then gives an alternative representation for any scalar of scaling dimension Δ via

$$\Phi(X) = |X|^{-\Delta}\int_{(\mathbb{CP}^3_X)^\vee} \mathrm{D}^3Z \wedge \tilde{f}|_X\,, \qquad \tilde{f} \in H^{0,3}(Q,\mathcal{O}(\Delta-4,-\Delta))\,. \tag{3.26}$$

More non-trivial is the *indirect Penrose transform*, which describes AdS scalars by elements of an entirely different cohomology group:

$$H^{0,2}(Q,\mathcal{O}(1-\Delta,\Delta-3)) \cong \{\Phi(X) \text{ on } \mathrm{AdS}_5 \,|\, \Box_{\mathrm{AdS}}\Phi = \Delta(\Delta-4)\Phi\}\,. \tag{3.27}$$

The existence of this alternative description for a scalar Φ is related to a certain obstruction problem in twistor space [17, 75, 90]. In particular, allowing g to extend off the quadric $Z \cdot W = 0$ in $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$ relates g to a direct Penrose transform representative by

$$\bar{\partial}g = (Z\cdot W)\,f\,, \tag{3.28}$$

where f takes values in $H^{0,3}(Q,\mathcal{O}(-\Delta,\Delta-4))$.

Equation (3.28) can be used to produce an integral formula for Φ in terms of g (this is adapted from a similar argument for the indirect transform for flat 6-dimensional space-time [75]). Extending off the quadric is accomplished at the level of the incidence relations by imposing

$$Z^A = X^{AB}\left(W_B + \delta W_B\right), \tag{3.29}$$

for some 'small' δW_B and then considering the limit as $\delta W_B \to 0$. The space-time scalar is then defined in terms of g by:

$$\Phi(X) = |X|^{\Delta} \lim_{\delta W\to 0} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge \left(\frac{\bar{\partial} g}{Z\cdot W}\right)\Bigg|_{Z^A = X^{AB}(W_B+\delta W_B)} . \tag{3.30}$$

Note that 'dual' representatives for the indirect Penrose transform are also constructed using the inverted incidence relations (3.25); this amounts to describing Φ by $\tilde{g} \in H^{0,2}(Q,\mathcal{O}(\Delta-3,1-\Delta))$ in the obvious way.

3.3.2 Spinors: direct and indirect transform

A generic eight-component spinor field on AdS_5 can be separated into a chiral and anti-chiral parts, taking values in $\mathbb{S}_A$ or $\mathbb{S}^A$, respectively. Without loss of generality, consider those components with a downstairs spinor index, of the form $\Psi_A(X)$. The equation of motion for a chiral spinor in the projective description of AdS_5 is:

$$(\not{\mathbb{D}}\Psi)^B = (\Delta-2)\,\frac{X^{AB}}{|X|}\Psi_A\,, \tag{3.31}$$

where the relation $|m| = \Delta - 2$ has been used. As with the scalar wave equation, the Dirac equation can be reduced to a simpler form

$$\partial^{AB}\left(|X|^{-\Delta-\frac{1}{2}}\Psi_A\right) = 0, \tag{3.32}$$

together with $X\cdot\partial\Psi_A = 0$. Details of reduction are given along with those for the scalar in the appendix.

On the twistor space Q, these fields are described by a $(0,3)$-form with values in $\mathcal{O}(-\Delta-\frac{1}{2},\Delta-\frac{9}{2})$, for fixed Δ. This bundle is only well defined for $\Delta \in \mathbb{Z}+\frac{1}{2}$, which is again consistent with the expected conformal dimensions of spinor primary operators on the boundary. For $\psi \in \Omega^{0,3}(Q, \mathcal{O}(-\Delta - \frac{1}{2}, \Delta - \frac{9}{2}))$ we form a space-time spinor field as

$$\Psi_A(X) = |X|^{\Delta+\frac{1}{2}} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge W_A \, \psi|_X \,, \tag{3.33}$$

where again $\psi|_X$ denotes that the incidence relations have been imposed. It is straightforward to show that the equation of motion (3.32) holds for Ψ_A if and only if $\bar{\partial}\psi = 0$. Once more, a homological argument demonstrates that every chiral, massive spinor on AdS_5 can be represented by (3.33) for some choice of ψ in the relevant cohomology [17, 14]. This gives the *direct* Penrose transform for spinors:

$$H^{0,3}\left(Q, \mathcal{O}\left(-\Delta - \frac{1}{2}, \Delta - \frac{9}{2}\right)\right) \cong \left\{\Psi_A(X) \text{ on } \mathrm{AdS}_5 \,|\, (\not{D}\Psi)^B = \Delta \frac{X^{AB}}{|X|} \Psi_A\right\}. \tag{3.34}$$

The integral formula (3.33) realizes this isomorphism from the left to the right.

Just like in the case of the scalar, there is an *indirect* version of the Penrose transform for spinors, given by

$$H^{0,2}\left(Q, \mathcal{O}\left(\frac{3}{2} - \Delta, \Delta - \frac{5}{2}\right)\right) \cong \left\{\Psi_A(X) \text{ on } \mathrm{AdS}_5 \,|\, (\not{D}\Psi)^B = \Delta \frac{X^{AB}}{|X|} \Psi_A\right\}. \tag{3.35}$$

The existence of an indirect transform is again related to an obstruction problem in twistor space [17, 75], with any indirect representative χ related to a direct representative ψ by

$$\bar{\partial}\chi = (Z \cdot W)^2 \psi \,. \tag{3.36}$$

Using this, an integral formula for the indirect transform is given by extending off the quadric in a similar fashion to the scalar:

$$\Psi_A(X) = |X|^{\Delta+\frac{1}{2}} \lim_{\delta W \to 0} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge W_A \left(\frac{\bar{\partial}\chi}{(Z \cdot W)^2}\right)\bigg|_{Z^A = X^{AB}(W_B + \delta W_B)} . \tag{3.37}$$

Note that there are 'dual' versions of both the direct and indirect transform; both are given by swapping the weights of Z and W, corresponding to the inverse incidence relations (3.25).

3.4 Free theory, Bulk-to-boundary propagators & 2-point functions

In applications of twistor theory to Minkowski space, the Penrose transform can be used to encode physically relevant external states in terms of twistor data. A basic example relevant for scattering amplitude calculations is a momentum eigenstate: an on-shell space-time field modelled on $\mathrm{e}^{ik\cdot x}$ is encoded in terms of certain distributional cohomology classes on twistor space [5, 75]. In AdS, the S-matrix is replaced by correlation functions of specified boundary data for the space-time fields [57, 114]. In this setup the appropriate external states are bulk-to-boundary propagators that propagate the boundary data into the AdS bulk. Computing the tree-level n-point correlation functions in the bulk boils down to extracting that piece of the classical generating functional which is multilinear in these external states on the AdS background.

In this section, we demonstrate that the most basic part of this AdS/CFT dictionary can be translated to twistor space by giving explicit representatives for scalar and spinor bulk-to-boundary propagators. The two-point functions for these fields are then derived in a purely twistorial manner by writing the free bulk theory in twistor variables.

3.4.1 Scalars

Holomorphic, first-order action functionals present a natural candidate for describing free theories on twistor space. For direct representatives, such an action is simply:

$$S[f,h] = \int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \bar{\delta}(Z\cdot W) \wedge h \wedge \bar{\partial} f\,, \tag{3.38}$$

where the top-degree holomorphic form on Q is written as

$$\int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \bar{\delta}(Z\cdot W) = \oint \frac{\mathrm{D}^3 Z \wedge \mathrm{D}^3 W}{Z\cdot W}\,,$$

with the holomorphic delta function $\bar{\delta}(Z\cdot W)$ equivalent to a contour integral localizing the measure to the quadric $Z\cdot W = 0$ inside $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$. This measure is a $(5,0)$-form on Q valued in $\mathcal{O}(3,3)$.

This action is a functional of $f \in \Omega^{0,3}(Q, \mathcal{O}(-\Delta, \Delta-4))$ and $h \in \Omega^{0,1}(Q, \mathcal{O}(\Delta-3, 1-\Delta))$ and its field equations are simply

$$\bar{\partial} f = 0 = \bar{\partial} h\,,$$

imposing that f and h are cohomology classes on-shell. By (3.24), it follows that $\bar{\partial} f = 0$ corresponds to the equation of motion $\Box_{\mathrm{AdS}}\Phi = \Delta(\Delta-4)\Phi$ for a scalar field. The second field equation, $\bar{\partial} h = 0$, is actually non-dynamical on space-time, as the cohomology group $H^{0,1}(Q,\mathcal{O}(\Delta-3,1-\Delta))$ is empty [17, 14]. Hence, h is just a Lagrange multiplier and solutions to the field equations are in one-to-one correspondence with solutions to the massive scalar equation of motion on AdS_5.

It is easy to see that the action (3.38) is not suitable for computing any observables in the bulk theory, though. Indeed, the action vanishes when evaluated on solutions to the equations of motion, whereas the appropriate space-time action is equal to a boundary term when evaluated on extrema. So although (3.38) gives the correct equations of motion, it is *not* equivalent to the free space-time action. This is analogous to the difference between space-time actions with kinetic terms $\partial\Phi\cdot\partial\Phi$ and $\Phi\Box\Phi$: they have the same equations of motion, although the former is equal to a boundary term on-shell whereas the latter vanishes.

To write a twistor action with non-vanishing extrema, the variational problem must involve an indirect representative $g \in \Omega^{0,2}(Q,\mathcal{O}(1-\Delta,\Delta-3))$ and its dual $\tilde{g} \in \Omega^{0,2}(Q,\mathcal{O}(\Delta-3,1-\Delta))$ coupled to fixed 'sources' in the twistor space. For a given Δ these sources are specified by a cohomology class $f \in H^{0,3}(Q,\mathcal{O}(-\Delta,\Delta-4))$ and its dual $\tilde{f} \in H^{0,3}(Q,\mathcal{O}(\Delta-4,-\Delta))$ which are *not* part of the variational problem. The action is:

$$S[g,\tilde{g}] = \int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \left[\bar{\delta}'(Z\cdot W)\wedge\tilde{g}\wedge\bar{\partial}g - \bar{\delta}(Z\cdot W)\wedge f\wedge\tilde{g} + \bar{\delta}(Z\cdot W)\wedge\tilde{f}\wedge g\right], \quad (3.39)$$

where $\bar{\delta}'(Z\cdot W) = \bar{\partial}(Z\cdot W)^{-2}$ is the $(0,1)$-distribution which acts like a derivative of a delta function. Since $f,\tilde{f}$ themselves constitute direct Penrose transform representatives, this is not the usual picture one has for physical sources. Instead, one should view $f,\tilde{f}$ as arising from an auxiliary variational problem, akin to the action (3.38).

The equations of motion arising from (3.39) are

$$\bar{\delta}'(Z\cdot W)\,\bar{\partial}g = \bar{\delta}(Z\cdot W)\,f \quad \Leftrightarrow \quad \bar{\partial}g = (Z\cdot W)\,f, \quad (3.40)$$
$$\bar{\delta}'(Z\cdot W)\,\bar{\partial}\tilde{g} = \bar{\delta}(Z\cdot W)\,\tilde{f} \quad \Leftrightarrow \quad \bar{\partial}\tilde{g} = (Z\cdot W)\,\tilde{f},$$

which are precisely the correct on-shell conditions (3.28) for indirect Penrose transform representatives. This refined action is non-vanishing when evaluated on solutions to these equations of motion:

$$S[g,\tilde{g}]\big|_{\mathrm{on-shell}} = \int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \bar{\delta}'(Z\cdot W)\wedge\tilde{g}\wedge\bar{\partial}g\,. \quad (3.41)$$

So although (3.39) requires the addition of source terms, it leads to sensible equations of motion and is non-zero when evaluated on extrema, making it a good candidate for computing AdS$_5$ observables in twistor space. The two-point function should be given by (3.41), where $g, \tilde{g}$ are chosen to represent the external states: bulk-to-boundary propagators.

For a massive scalar on AdS$_5$, the bulk-to-boundary propagator K_Δ is a solution to the equation of motion which becomes proportional to a delta function on the boundary. In Poincaré coordinates, these conditions read:

$$\Box_{\mathrm{AdS}} K_\Delta(r,x;y) = \Delta(\Delta-4)\, K_\Delta(r,x;y)\,, \qquad \lim_{r\to 0} r^{\Delta-4} K_\Delta(r,x;y) = \delta^4(x-y)\,,$$

where $(r, x^{\alpha\dot{\alpha}})$ is a bulk point in AdS$_5$, and $y^{\alpha\dot{\alpha}}$ is a point on the boundary S^4. An expression for this bulk-to-boundary propagator is given in Poincaré coordinates by

$$K_\Delta(r,x;y) = \mathrm{c}_\Delta \left(\frac{r}{r^2+(x-y)^2}\right)^{\Delta}\,, \tag{3.42}$$

where c_Δ is an overall normalization which will be ignored from now on.

How is (3.42) presented on twistor space? Since K_Δ is a solution to the equation of motion, it should be representable by the Penrose transform. Consider the distributional form

$$f_\Delta(Z,W) = [AB]^{\Delta}\, \frac{\bar{\delta}^3_{\Delta-4}(W,A)}{(Z\cdot B)^{\Delta}}\,, \tag{3.43}$$

where A_A, B_A are two fixed points in $\mathbb{CP}^3$, and $[AB] = I^{CD} A_C B_D$ denotes the contraction of A and B with the infinity twistor of the boundary. The delta function $\bar{\delta}^3_{\Delta-4}(W,A)$ is defined as

$$\bar{\delta}^3_{\Delta-4}(W,A) = \int \frac{\mathrm{d}t}{t}\, t^{\Delta} \bigwedge_{A=1}^{4} \bar{\partial}\left(\frac{1}{W_A + t\, A_A}\right).$$

This gives a $(0,3)$-form distribution enforcing the projective coincidence of its two arguments which is homogeneous of degree $\Delta-4$ in W and $-\Delta$ in A.

Up to singularities determined entirely by the fixed points A, B, this object is $\bar{\partial}$-closed and is homogeneous of degree zero in A, B. Thus, (3.43) can be treated as a class in $H^{0,3}(Q, \mathcal{O}(-\Delta, \Delta-4))$, so the direct Penrose transform can be applied to give a space-time field

$$|X|^{\Delta} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge \frac{\bar{\delta}^3_{\Delta-4}(W,A)}{(X^{CD} W_D B_C)^{\Delta}}\, [AB]^{\Delta} = \frac{|X|^{\Delta}[AB]^{\Delta}}{(X^{CD} A_C B_D)^{\Delta}}\,.$$

Notice that A and B only appear as the skew-symmetric combination $Y_{CD} = A_{[C}B_{D]}$ through $[AB]$, $X^{CD}A_C B_D$ in the final answer. Since $Y^2 = 0$, this corresponds to a fixed point on the boundary of AdS_5, so:

$$\int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge f_\Delta|_X = \frac{|X|^\Delta (I\cdot Y)^\Delta}{(X\cdot Y)^\Delta}. \tag{3.44}$$

It is easy to confirm (by going to the Poincaré parametrization, for instance) that this expression is equal to (3.42). Thus, (3.43) is a direct transform representative for the scalar bulk-to-boundary propagator.

An indirect representative for the bulk-to-boundary propagator is given by

$$g_\Delta(Z,W) = [AB]^\Delta \int s^{\Delta-1}\mathrm{d}s\,\frac{\bar{\delta}^3_{\Delta-3}(W,A(s))}{(Z\cdot A)^{\Delta-1}}, \tag{3.45}$$

where $A(s) = A + sB$ parametrizes a point on the projective line spanned by $A\wedge B$ in $\mathbb{CP}^3$. The integral over the parameter s reduces the distributional form degree of g to $(0,2)$, and it is easy to show that (3.45) is homogeneous of degree $1-\Delta$ in Z and $\Delta-3$ in W. Note that g is not obviously $\bar{\partial}$-closed, as

$$\bar{\partial} g_\Delta = [AB]^\Delta \int s^{\Delta-1}\mathrm{d}s\,\bar{\delta}^{(\Delta-2)}(Z\cdot A)\,\bar{\delta}^3_{\Delta-3}(W,A(s)), \tag{3.46}$$

where $\bar{\delta}^{(\Delta-2)}(Z\cdot A)$ is a $(0,1)$-distribution acting like the $(\Delta-2)^{\mathrm{th}}$-derivative of a delta-function:

$$\bar{\delta}^{(\Delta-2)}(Z\cdot A) := \bar{\partial}\left(\frac{1}{(Z\cdot A)^{\Delta-1}}\right).$$

However, by integrating (3.46) against test functions it can be shown that $\bar{\partial} g_\Delta = 0$ as a distribution on Q and that furthermore $\bar{\partial} g_\Delta = (Z\cdot W) f$ when extended off the quadric in accordance with (3.28). This representative can be evaluated to a space-time field using the integral formula (3.30):

$$\begin{aligned}
&|X|^\Delta \lim_{\delta W\to 0} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge \left(\frac{\bar{\partial} g_\Delta}{Z\cdot W}\right)\bigg|_{Z^A = X^{AB}(W_B+\delta W_B)} \\
&= \frac{(I\cdot Y)^\Delta |X|^\Delta}{(X\cdot Y)^{\Delta-1}} \lim_{\delta W\to 0}\int \frac{s^{\Delta-1}\,\mathrm{d}s}{X^{AB}A_A\delta W_B + sX^{AB}B_A\delta W_B}\,\bar{\delta}^{(\Delta-2)}\left(s + \frac{X^{AB}A_A\delta W_B}{X\cdot Y}\right) \\
&\qquad = \frac{(I\cdot Y)^\Delta |X|^\Delta}{(X\cdot Y)^\Delta} \lim_{\delta W\to 0}\left[\frac{X\cdot Y}{X\cdot Y - X^{AB}B_A\delta W_B} + O(\delta W)\right] = \frac{(I\cdot Y)^\Delta |X|^\Delta}{(X\cdot Y)^\Delta},
\end{aligned}$$

which is again the correct bulk-to-boundary propagator.

In space-time, evaluating the quadratic action on bulk-to-boundary propagators gives the AdS two-point function, equal to the two-point function of local operators of conformal dimension Δ in a CFT living on the boundary. This calculation was one of the first tests of the AdS/CFT correspondence [57, 114], and consequently gives a important check for the twistor formalism. On-shell, the free twistor action reduces to (3.41), now evaluated on

$$\int \mathrm{D}^3 Z\, \mathrm{D}^3 W\, \bar{\delta}'(Z\cdot W)\, \tilde{g}_\Delta \wedge \bar{\partial} g_{\Delta'}\,,$$

with $\tilde{g}_\Delta, g_{\Delta'}$ of the form (3.45) and distinct boundary points. The $\mathrm{D}^3 Z$ and $\mathrm{D}^3 W$ integrals in this pairing can be evaluated straightforwardly to give:

$$\begin{aligned}
(I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta'} &\int \mathrm{D}^3 Z\, \mathrm{D}^3 W\, \bar{\delta}'(Z\cdot W)\,\frac{s^{\Delta-1}\mathrm{d}s}{(W\cdot A)^{\Delta-1}}\,\bar{\delta}^3(Z, A(s)) \\
&\qquad \times t^{\Delta'-1}\mathrm{d}t\,\bar{\delta}^{(\Delta'-2)}(Z\cdot C)\,\bar{\delta}^3(W, C(t)) \\
&= (I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta'}\int s^{\Delta-1}\mathrm{d}s\, t^{\Delta'-1}\mathrm{d}t\,\frac{\bar{\delta}'(A(s)\cdot C(t))}{(A\cdot C(t))^{\Delta-1}}\,\bar{\delta}^{(\Delta'-2)}(A(s)\cdot C)\,, \quad (3.47)
\end{aligned}$$

where $Y_1^{AB} = A^{[A}B^{B]}$, $Y_{2AB} = C_{[A}D_{B]}$, $A(s) = A + sB$, and $C(t) = C + tD$. Note that the expression is projectively well-defined only if the two scaling dimensions are equal, so we set $\Delta = \Delta'$.

The scaling and distributional properties of the remaining portions of the integrand allow the s and t integrals to be performed in a basically algebraic manner. It is straightforward to show that (3.47) is equal to

$$\begin{aligned}
(I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta} &\int \frac{s^{\Delta-1}\mathrm{d}s\, t^{\Delta-1}\mathrm{d}t}{(B\cdot C)^{\Delta-1}(A\cdot C(t))^{\Delta-1}}\,\bar{\delta}'(A(s)\cdot C(t))\,\bar{\delta}^{(\Delta-2)}\left(\frac{A\cdot C}{B\cdot C}+s\right) \\
&= (I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta}\int \frac{t^{\Delta-1}\mathrm{d}t}{(A\cdot C(t))^{\Delta-1}}\,\frac{(A\cdot C)^{\Delta-1}}{(B\cdot C)^{\Delta}}\,\bar{\delta}^{(\Delta-1)}\left(t\left(A\cdot D - \frac{A\cdot C}{B\cdot C}B\cdot D\right)\right) \\
&= (I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta}\int \frac{t^{\Delta-1}\mathrm{d}t}{(A\cdot C(t))^{\Delta-1}}\,\frac{(A\cdot C)^{\Delta-1}}{(A\cdot D B\cdot C - A\cdot C B\cdot D)^{\Delta}}\,\bar{\delta}^{(\Delta-1)}(t) \\
&= \frac{(I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta}}{(A\cdot D B\cdot C - A\cdot C B\cdot D)^{\Delta}} = \frac{(I\cdot Y_1)^{\Delta}\,(I\cdot Y_2)^{\Delta}}{(Y_1\cdot Y_2)^{\Delta}}\,. \quad (3.48)
\end{aligned}$$

This is precisely the desired form of the 2-point function for massive scalars in AdS$_5$. Written more compactly,

$$\int \mathrm{D}^3 Z\, \mathrm{D}^3 W\, \bar{\delta}'(Z\cdot W)\, \tilde{g}_\Delta \wedge \bar{\partial} g_{\Delta'} = \frac{\delta_{\Delta\Delta'}}{(y_1 - y_2)^{2\Delta}}\,, \tag{3.49}$$

where $y_1, y_2 \in S^4$ lie on the boundary. As expected, this is the two-point function $\langle \mathcal{O}_\Delta(y_1)\mathcal{O}_{\Delta'}(y_2)\rangle$ of local operators in any four-dimensional CFT.

In the larger context of AdS/CFT the bulk partition function of a scalar field with boundary value ϕ is equivalent to a generating functional,

$$\left\langle \exp\left(\int_{S^4} \mathrm{d}^4 y\, \phi(y)\mathcal{O}_\Delta(y)\right)\right\rangle_{\mathrm{CFT}_4},$$

where $\mathcal{O}_\Delta$ is a local operator in the dual CFT of conformal dimension Δ. The calculation of (3.49) demonstrates that the quadratic portion of this functional can be obtained from the twistor space of the AdS_5 bulk.

3.4.2 Fundamental solution to the AdS wave equation

It is also possible to write the fundamental solution (Green's function) to the wave equation in AdS_5 in twistor space. In spacetime, the solution is given by

$$\phi(X,Y) = \xi^\Delta {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta-1; -\xi^2\right) \tag{3.50}$$

where ${}_2F_1$ is Gaussian hypergeometric function and

$$\xi = \frac{X\cdot Y}{|X||Y|}. \tag{3.51}$$

We show that the Penrose transform of

$$f(Z,W) = |Y|^{4-\Delta}\int \frac{dt}{t}\frac{ds}{s} t^{4-\Delta}\bar{\delta}^4(Z^A - tY^{AB}W_B - sZ_*^A) \tag{3.52}$$

depends only on X and Y through ξ. The AdS wave equation reduces to a second order ODE on functions of ξ only, and its two solutions are the fundamental solutions corresponding to the two choices of Δ satisfying $m^2 = \Delta(\Delta-4)$. Since a Penrose transform must automatically satisfy the AdS wave equation, it must be Penrose representative for the fundamental solution for one of these choices of Δ.

The Penrose transform of $f(Z,W)$ does not depend on the reference twistor Z_* since a change of Z_* can be accommodated by a shift of W_A. It is

$$|X|^\Delta \int D^3W f(XW,W) = |X|^\Delta |Y|^{4-\Delta} \int D^3W \frac{dt}{t}\frac{ds}{s} t^{4-\Delta} \bar{\delta}^4((X^{AB}-tY^{AB})W_B - sZ_*^A) \tag{3.53}$$

$$= |X|^\Delta |Y|^{4-\Delta} \int D^3W \frac{dt}{t} t^{4-\Delta} \bar{\delta}^3_{-4,0}((X^{AB}-tY^{AB})W_B, Z_*^A) \tag{3.54}$$

$$= |X|^\Delta |Y|^{4-\Delta} \int \frac{dt}{t} \frac{t^{4-\Delta}}{(X-tY)^4} \tag{3.55}$$

$$= |X|^\Delta |Y|^{4-\Delta} \int \frac{dt}{t} \frac{t^{4-\Delta}}{(X^2 - 2X\cdot Y t + t^2 Y^2)^2} \tag{3.56}$$

$$= |X|^\Delta \int \frac{dt}{t} \frac{t^{4-\Delta}}{(X^2 - \frac{2X\cdot Y}{|Y|} t + t^2)^2} \tag{3.57}$$

$$= \int \frac{dt}{t} \frac{t^{4-\Delta}}{(1-\xi t + t^2)^2} \tag{3.58}$$

which depends only on ξ.

3.4.3 Spinors

In contrast to the scalar, the standard space-time action for a free AdS spinor of mass m vanishes on-shell. To obtain non-trivial two-point functions, a boundary term which respects the AdS isometries and does not alter equations of motion must be added to the action [58]. In twistor space, the free action is given by generalizing that of the scalar:

$$S[\chi,\tilde{\chi}] = \int \mathrm{D}^3Z \wedge \mathrm{D}^3W \wedge \left[\bar{\delta}''(Z\cdot W)\wedge\tilde{\chi}\wedge\bar{\partial}\chi - \bar{\delta}(Z\cdot W)\wedge\psi\wedge\tilde{\chi} + \bar{\delta}(Z\cdot W)\wedge\tilde{\psi}\wedge\chi\right], \tag{3.59}$$

where the variational problem involves the off-shell fields $\chi \in \Omega^{0,2}(Q,\mathcal{O}(\frac{3}{2}-\Delta,\Delta-\frac{5}{2}))$, $\tilde{\chi} \in \Omega^{0,2}(Q,\mathcal{O}(\Delta-\frac{5}{2},\frac{3}{2}-\Delta))$, while $\psi \in H^{0,3}(Q,\mathcal{O}(-\Delta-\frac{1}{2},\Delta-\frac{9}{2}))$ and $\tilde{\psi} \in H^{0,3}(Q,\mathcal{O}(\Delta-\frac{9}{2},-\Delta-\frac{1}{2}))$ are treated as fixed 'sources.' The equations of motion are easily seen to coincide with (3.36) for indirect representatives:

$$\bar{\delta}''(Z\cdot W)\,\bar{\partial}\chi = \bar{\delta}(Z\cdot W)\,\psi \quad\Leftrightarrow\quad \bar{\partial}\chi = (Z\cdot W)^2\,\psi\,, \tag{3.60}$$

$$\bar{\delta}''(Z\cdot W)\,\bar{\partial}\tilde{\chi} = \bar{\delta}(Z\cdot W)\,\tilde{\psi} \quad\Leftrightarrow\quad \bar{\partial}\tilde{\chi} = (Z\cdot W)^2\,\tilde{\psi}\,,$$

and the action evaluated on extrema is non-vanishing:

$$S[\chi,\tilde{\chi}]\big|_{\text{on-shell}} = \int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \bar{\delta}''(Z\cdot W) \wedge \tilde{\chi} \wedge \bar{\partial}\chi\,. \tag{3.61}$$

As in the case of the scalar, the on-shell sources $\psi, \tilde{\psi}$ should be viewed as arising from a separate variational problem.

Bulk-to-boundary propagators for spinor fields with scaling dimension Δ are given in twistor space by modifying those used for the scalar. In particular, a spinor bulk-to-boundary propagator is a solution to the free equation of motion, $K^{\bullet}_{\Delta\,A}(r,x;y)$, where $\bullet$ stands for a boundary Weyl spinor index (*i.e.*, a dotted or un-dotted SL$(2,\mathbb{C})$ index). The boundary spinor structure is encoded in twistor space by using the boundary infinity twistor (3.6) or $I_{AB} = \frac{1}{2}\epsilon_{ABCD}I^{CD}$. For instance, a direct representative with a dotted boundary index, say $\dot{\beta}$, reads:

$$\psi^{\dot{\beta}}_{\Delta}(Z,W) = [AB]^{\Delta-\frac{1}{2}} I^{BC} B_C \frac{\bar{\delta}^3_{\Delta-\frac{9}{2}}(W,A)}{(Z\cdot B)^{\Delta+\frac{1}{2}}} + [BA]^{\Delta-\frac{1}{2}} I^{BC} A_C \frac{\bar{\delta}^3_{\Delta-\frac{9}{2}}(W,B)}{(Z\cdot A)^{\Delta+\frac{1}{2}}} \tag{3.62}$$

Feeding this representative into the integral transform (3.33) gives a space-time formula

$$|X|^{\Delta+\frac{1}{2}} \frac{(I\cdot Y)^{\Delta-\frac{1}{2}}}{(X\cdot Y)^{\Delta+\frac{1}{2}}} I^{BC} Y_{CA} = \frac{r^{\Delta+\frac{1}{2}}}{(r^2+(x-y)^2)^{\Delta+\frac{1}{2}}} \begin{pmatrix} -\delta^{\dot{\beta}}_{\dot{\alpha}} & y^{\dot{\beta}\delta} \\ 0 & 0 \end{pmatrix},$$

where $Y_{AB} = A_{[A}B_{B]}$ is the boundary point. This expression can be made equal to the standard formula for the spinor bulk-to-boundary propagator [58, 78] after performing normalized gamma matrix contractions. An un-dotted boundary spinor index is given by taking the dual of (3.62) and replacing I^{AB} with I_{AB} in direct analogy to the scalar case.

An indirect representative for the bulk-to-boundary propagator is given by:

$$\chi^{\dot{\beta}}_{\Delta}(Z,W) = I^{BC} \int s^{\Delta-\frac{1}{2}} \mathrm{d}s \left([AB]^{\Delta-\frac{1}{2}} B_C \frac{\bar{\delta}^3_{\Delta-\frac{5}{2}}(W,A(s))}{(Z\cdot A)^{\Delta-\frac{3}{2}}} \right.$$
$$\left. + [BA]^{\Delta-\frac{1}{2}} A_C \frac{\bar{\delta}^3_{\Delta-\frac{5}{2}}(W,B(s))}{(Z\cdot B)^{\Delta-\frac{3}{2}}} \right), \tag{3.63}$$

with $A(s) = A + sB$ and $B(s) = B + sA$ parametrizing points on the line spanned by A, B. As in the case of the scalar, this representative is not obviously $\bar{\partial}$-closed:

$$\bar{\partial}\chi_{\Delta}^{\dot{\beta}} = I^{BC} \int s^{\Delta-\frac{1}{2}} \mathrm{d}s \Big([AB]^{\Delta-\frac{1}{2}} B_C \,\bar{\delta}^{(\Delta-\frac{5}{2})}(Z\cdot A)\, \bar{\delta}^{3}_{\Delta-\frac{5}{2}}(W, A(s)) + [BA]^{\Delta-\frac{1}{2}} A_C \,\bar{\delta}^{(\Delta-\frac{5}{2})}(Z\cdot B)\, \bar{\delta}^{3}_{\Delta-\frac{5}{2}}(W, B(s)) \Big) . \quad (3.64)$$

Direct calculation nevertheless shows that $\bar{\partial}\chi_{\Delta}^{\dot{\beta}} = 0$ as a distribution on Q, and furthermore that

$$|X|^{\Delta+\frac{1}{2}} \lim_{\delta W\to 0} \int_{\mathbb{CP}^3_X} \mathrm{D}^3 W \wedge W_A \left(\frac{\bar{\partial}\chi_{\Delta}^{\dot{\beta}}}{(Z\cdot W)^2} \right)\Bigg|_{Z^A = X^{AB}(W_B + \delta W_B)} = |X|^{\Delta+\frac{1}{2}} \frac{(I\cdot Y)^{\Delta-\frac{1}{2}}}{(X\cdot Y)^{\Delta+\frac{1}{2}}} I^{BC} Y_{CA} , \quad (3.65)$$

as desired.

To compute the two-point function in twistor space, the on-shell action (3.61) is evaluated on bulk-to-boundary representatives. It is straightforward to see that the result is only non-vanishing if one of the representatives has a dotted boundary index and the other has an un-dotted boundary index. Since these representatives contain two terms each, the total integrand of (3.61) will have four terms. One of these is given by

$$I_{AC} I^{BD} B_D D^C [AB]^{\Delta+\frac{1}{2}} \langle CD\rangle^{\Delta+\frac{1}{2}} \int \mathrm{D}^3 Z\, \mathrm{D}^3 W\, \bar{\delta}''(Z\cdot W)\, s^{\Delta-\frac{1}{2}} \mathrm{d}s\, t^{\Delta-\frac{1}{2}} \mathrm{d}t \, \frac{\bar{\delta}^3(Z, C(s))}{(W\cdot C)^{\Delta-\frac{3}{2}}} \bar{\delta}^{(\Delta-\frac{5}{2})}(Z\cdot A)\, \bar{\delta}^3(W, A(t)) ,$$

where $\langle CD\rangle = I_{AB} C^A D^B$. Each of the other three terms takes a similar form. All of the integrals in this expression can be evaluated against the distributional delta functions to give

$$
\begin{aligned}
&I_{AC} I^{BD} B_D D^C [AB]^{\Delta+\frac{1}{2}} \langle CD\rangle^{\Delta+\frac{1}{2}} \int s^{\Delta-\frac{1}{2}} \mathrm{d}s\, t^{\Delta-\frac{1}{2}} \mathrm{d}t\, \bar{\delta}''(C(s)\cdot A(t))\, \frac{\bar{\delta}^{(\Delta-\frac{5}{2})}(C(s)\cdot A)}{(A(t)\cdot C)^{\Delta-\frac{3}{2}}} \\
&\qquad = I_{AC} I^{BD} [AB]^{\Delta+\frac{1}{2}} \langle CD\rangle^{\Delta+\frac{1}{2}} \frac{B_D D^C (A\cdot C)^{\Delta-\frac{1}{2}}}{(Y_1\cdot Y_2)^{\Delta+\frac{1}{2}}} \int \frac{t^{\Delta-\frac{1}{2}} \mathrm{d}t}{(A(t)\cdot C)^{\Delta-\frac{3}{2}}}\, \bar{\delta}^{(\Delta-\frac{1}{2})}(t) \\
&\qquad = I_{AC} I^{BD} [AB]^{\Delta+\frac{1}{2}} \langle CD\rangle^{\Delta+\frac{1}{2}} \frac{(A\cdot C)\, B_D D^C}{(Y_1\cdot Y_2)^{\Delta+\frac{1}{2}}}\,,
\end{aligned}
$$

where $Y_{1\,AB} = A_{[A}B_{B]}$ and $Y_2^{AB} = C^{[A}D^{B]}$ are the two distinct boundary points.

Upon combining this expression with the results from the three other terms, one obtains

$$
\begin{aligned}
\int \mathrm{D}^3 Z \wedge \mathrm{D}^3 W \wedge \bar{\delta}''(Z\cdot W) \wedge \tilde{\chi}^{\alpha}_{\Delta} \wedge \bar{\partial}\chi^{\dot{\beta}}_{\Delta} &= (I\cdot Y_1)^{\Delta-\frac{1}{2}} (I\cdot Y_2)^{\Delta-\frac{1}{2}} \frac{I^{BD} Y_{1\,DE} Y_2^{EC} I_{AC}}{(Y_1\cdot Y_2)^{\Delta+\frac{1}{2}}} \\
&= \frac{(I\cdot Y_1)^{\Delta-\frac{1}{2}} (I\cdot Y_2)^{\Delta-\frac{1}{2}}}{(Y_1\cdot Y_2)^{\Delta+\frac{1}{2}}} \begin{pmatrix} 0 & 0 \\ (y_1-y_2)^{\alpha\dot{\beta}} & 0 \end{pmatrix}, \qquad (3.66)
\end{aligned}
$$

which in Poincaré coordinates is equivalent to the expected two-point function of spinor operators in a four-dimensional CFT:

$$
\left\langle j^{\alpha}_{\Delta}(y_1)\, j^{\dot{\beta}}_{\Delta'}(y_2) \right\rangle_{\mathrm{CFT}_4} = \delta_{\Delta\Delta'} \frac{(y_1-y_2)^{\alpha\dot{\beta}}}{(y_1-y_2)^{2\Delta+1}}\,. \qquad (3.67)
$$

3.5 Discussion

In this chapter we have investigated the twistor space of AdS_5. In particular we constructed explicit twistor representatives for bulk-to-boundary propagators for fields of various spins, and verified that a natural twistor action for these fields reproduces the expected form of the two-point boundary correlation function.

It is worth noting that the bulk-to-boundary representatives and free twistor actions presented here can be adapted to AdS_3 using the language of 'minitwistors' [59, 67, 68]. The minitwistor space of AdS_3 is the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$ inside $\mathbb{CP}^3$, with space-time points corresponding to conics inside this quadric. Our direct bulk-to-boundary representatives are easily transcribed into the minitwistor Penrose transform, and two-point functions can be obtained analogously.

The situation is somewhat different in AdS_4, where the Penrose transform describes only conformally coupled bulk fields. Here, twistor methods have been applied in [96, 3], with the aim of finding compact expressions for tree-level bulk correlators, but it is not yet clear how to encode the external states in a useful way. We hope that the study of twistor theory and bulk observables in AdS_5 may clarify these issues in the AdS_4 setting.

Our further hope is that the results of this chapter, in particular the dual role of Q as both the twistor space of AdS_5 and the ambitwistor space of the boundary, can be used to shed light on the AdS/CFT correspondence from a twistor perspective. However, much work remains to be done. The construction of a non-linear theory on Q describing AdS supergravity remains a challenging problem.

Chapter 4

Minitwistors and 3d Yang-Mills-Higgs theory

4.1 Introduction

This chapter is concerned with perturbative aspects of Yang–Mills–Higgs theory

$$S_0[A,\Phi] = -\frac{1}{2g^2}\int_{\mathbb{R}^3} \mathrm{tr}\,(F\wedge *F + \mathrm{D}\Phi\wedge *\mathrm{D}\Phi)\,, \tag{4.1}$$

in three dimensions. In this action, $D = \mathrm{d} + A$ is the covariant derivative and F its curvature, the Higgs field Φ is a scalar in the adjoint of the gauge group, $*$ is the Hodge star on $\mathbb{R}^3$ with a flat Euclidean metric, and g is a coupling constant. Note that g^2 has mass dimension $+1$ in $d = 3$, so this theory is asymptotically free. This action is naturally interpreted as the dimensional reduction of pure Yang–Mills theory

$$S[A^{(4)}] = -\frac{1}{2g^2_{(4)}}\int_{\mathbb{R}^3\times S^1} \mathrm{tr}\left(F^{(4)}\wedge *F^{(4)}\right) \tag{4.2}$$

on $\mathbb{R}^3\times S^1$ in the limit that the radius of the circle shrinks to zero size, with the Higgs field emerging as the component of $A^{(4)}$ along the S^1 directions, and where $g^2_{(4)} = \mathrm{Vol}(S^1)\,g^2$.

Many properties of YMH_3 are inherited from this relationship with YM_4. For the purposes of this work, the key fact is that, just as it is possible to perturbatively expand YM_4 around the self–dual sector, so too YHM_3 admits a perturbative expansion around solutions of the Bogomolny equations [107]

$$*\,F = \mathrm{D}\Phi\,. \tag{4.3}$$

These equations are the dimensional reduction of the self–duality condition $F^{(4)} = *^{(4)} F^{(4)}$ describing instantons in $d = 4$. Solutions to (4.3) automatically solve the full field equations

$$\mathrm{D} * F = [*D\Phi, \Phi]\,, \qquad \mathrm{D} * \mathrm{D}\Phi = 0 \tag{4.4}$$

and Bianchi identity of (4.1).

To expand YMH_3 around solutions of the Bogomolny equations, consider the new action

$$\begin{aligned} S[A,\Phi] &= -\frac{1}{2g^2}\int_{\mathbb{R}^3} \mathrm{tr}\,[(F - *\mathrm{D}\Phi)\wedge(*F - \mathrm{D}\Phi)] \\ &= S_0[A,\Phi] + \frac{1}{g^2}\int_{\mathbb{R}^3} \mathrm{tr}\,(F\wedge D\Phi)\,. \end{aligned} \tag{4.5}$$

Using the Bianchi identity $DF = 0$, the final term can be written $\int_{\mathbb{R}^3} \mathrm{d}\,\mathrm{tr}(F\Phi)$ and so is a total derivative that does not affect perturbation theory[1] on $\mathbb{R}^3$. We now introduce a Lagrange multiplier B which is a 1-form valued in the adjoint of the gauge group. With this field, the action (4.5) can be written as:

$$S[A,B,\Phi] = \int_{\mathbb{R}^3} \mathrm{tr}\,[B\wedge(F - *\mathrm{D}\Phi)] + \frac{g^2}{2}\int_{\mathbb{R}^3} \mathrm{tr}\,(B\wedge *B)\,. \tag{4.6}$$

The equations of motion for this action are now

$$\mathrm{D}\Phi - *F = g^2 B\,, \qquad \text{and} \qquad \mathrm{D}B = - * [B,\Phi]\,, \tag{4.7}$$

and it is easy to see that integrating out B in the path integral results in the action (4.6).

The significance of rewriting YMH_3 in the form (4.6) is that the coupling g now acts as a parameter for expanding around the monopole sector. Indeed, when $g = 0$, (A,Φ) obey the Bogomolny equations while B acts as a linear *anti*-monopole gauge field propagating on the non-linear monopole background. For $g \neq 0$, the field configuration is deformed away from the monopole equations by $*B$. This construction is simply the dimensional reduction of the Chalmers-Siegel action [31] for YM_4, which gives a perturbative expansion around the self-dual sector.

The monopole sector of YMH_3 is classically integrable (see *e.g.* [80, 106, 36, 59, 60, 79, 22]). This suggests that it should be an attractive background around which to study perturbation theory, just as the MHV expansion of YM_4 allows us to systematically construct n-particle gluon amplitudes, allowing arbitrary numbers of positive helicity $4d$ gluons at no cost [30].

[1] In the presence of monopoles, this term is a topological invariant.

The aim of this chapter is to study this perturbation theory, constructing all n-particle tree amplitudes in the theory (4.6). As in $d = 4$, our approach will involve moving to twistor space where the integrability of the Bogomolny equations becomes manifest. For $\mathbb{R}^3$ the relevant twistor space is known as *minitwistor space*. This space was originally introduced by Hitchin in [59] with the aim of studying the Bogomolny equations (4.3) (rather than full YMH_3 theory) and has been extensively studied since. We begin in section 4.2 by giving a brief introduction to the geometry of minitwistor space, largely following the perspective of [67, 109]. We then review the Penrose transform and Hitchin–Ward correspondence, showing how solutions of the massless field equations and Bogomolny equations can be expressed in terms of cohomology classes and holomorphic vector bundles on minitwistor space. In section 4.3 we construct an action in minitwistor space that describes YMH_3 theory perturbatively. We explicitly demonstrate the off–shell equivalence of this action with (4.6), and also show how it may be understood as the dimensional reduction of the twistor action for YM_4 found in [74, 20]. We also consider the maximally supersymmetric version of the theory. Finally, we obtain a concise generating function for all tree–level amplitudes in (supersymmetric) YMH_3, written in terms of higher degree maps to minitwistor space. We show analytically that our expression for the amplitudes reproduces the correct 3–particle '$\overline{\text{MHV}}$' and n-particle 'MHV' amplitudes, and also that it factorizes correctly at all degrees. We explain that this generating function can be understood as the dimensional reduction of the RSVW expression [115, 89] for $\mathcal{N} = 4$ SYM in $d = 4$.

Various parts of our story have appeared before. In particular, the minitwistor action corresponding to the 'monopole' theory

$$S[A, \Phi, B] = \int_{\mathbb{R}^3} \text{tr}\,[B \wedge (F - *\text{D}\Phi)] \tag{4.8}$$

(*i.e.* the action (4.6) at $g = 0$) has appeared previously in [87], while expressions for the on–shell 'MHV' *amplitudes* appeared in [72]. However, the fact that the off–shell continuation of these amplitudes can be combined with the minitwistor action of [87] to give an action for full YMH_3 appears not to have been appreciated. Amplitudes for YMH_3 have been studied in [35, 10, 28]. In particular, in [28] Cachazo *et al.* gave a connected prescription formula for all tree amplitudes in YMH_3 that is equivalent to ours. However, the formula of [28] was simply the $\mathcal{N} = 4$ SYM formula together with a set of δ–functions enforcing that the external particles have no momentum in one of the four space–time directions. We show that the effects of these δ–functions can be incorporated into a concise expression that is inherently three–dimensional.

It is worth noting also that there is an alternative definition of twistors, given in [61], which are useful for describing amplitudes in 3d superconformal theoreies.

This chapter is based on collaborative work and primarily reproduced from [9].

4.2 Minitwistor Theory

For our purposes, *minitwistor space* will be the space of oriented geodesics in $\mathbb{R}^3$. In this section we review the geometry of minitwistor space and its supersymmetric generalisation, how free fields on $\mathbb{R}^3$ are encoded in minitwistor space, and the minitwistor description of the monopole sector.

4.2.1 Geometry of minitwistor space

Cartesian coordinates $\mathbf{x}$ on $\mathbb{R}^3$ may be encoded in a 2×2 symmetric matrix

$$x^{\alpha\beta} = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} -x+\mathrm{i}y & z \\ z & x+\mathrm{i}y \end{pmatrix} \tag{4.9}$$

where $\det x^{\alpha\beta} = \frac{1}{2}\mathbf{x}\cdot\mathbf{x}$ is (half) the Euclidean norm.

Minitwistor space $\mathbb{MT}$ is the total space of the holomorphic tangent bundle $T\mathbb{CP}^1$ to a Riemann sphere. As a line bundle, $T\mathbb{CP}^1 \cong \mathcal{O}(2)$, so we can describe $\mathbb{MT}$ using homogeneous coordinates $[u, \lambda_\alpha]$ where for $\alpha = 0, 1$, λ_α are homogeneous coordinates on the Riemann sphere $\mathbb{CP}^1$ and u is a coordinate along the fibres at each point. These coordinates are considered up to overall $\mathbb{C}^*$ rescalings acting as

$$(u, \lambda_\alpha) \sim (r^2 u, r\lambda_\alpha)\,, \tag{4.10}$$

for all $r \in \mathbb{C}^*$, and the two λs are never simultaneously zero.

The correspondence between $\mathbb{MT}$ and space–time is encoded in the *incidence relations*

$$u = x^{\alpha\beta}\lambda_\alpha\lambda_\beta\,, \tag{4.11}$$

where we temporarily allow the $x^{\alpha\beta}$ to be complex. For fixed $x^{\alpha\beta}$, equation (4.11) describes a section $u : \mathbb{CP}^1 \to T\mathbb{CP}^1$ so a point $x \in \mathbb{C}^3$ corresponds to a section of $\mathbb{MT}$ over $\mathbb{CP}^1$. We will call such sections *minitwistor lines.* Note that *any* two minitwistor lines $X, Y \subset \mathbb{MT}$ (defined by $u = x^{\alpha\beta}\lambda_\alpha\lambda_\beta$ and $u = y^{\alpha\beta}\lambda_\alpha\lambda_\beta$) will intersect each other in two points, since $(x-y)^{\alpha\beta}\lambda_\alpha\lambda_\beta = 0$ can be regarded as a quadratic equation in the local coordinate $z = \lambda_1/\lambda_0$. This reflects the fact that the normal bundle to each minitwistor line is $\mathcal{O}(2)$.

Suppose we label the two intersection points in $\mathbb{MT}$ by (u, λ_α) and (u', λ'_α). The incidence relations imply that the vector connecting the two points in $\mathbb{R}^3$ must take the form $(x-y)^{\alpha\beta} = \lambda^{(\alpha}\lambda'^{\beta)}$. Hitchin [59] defines a (holomorphic) conformal structure on $\mathbb{C}^3$ by declaring $x, y \in \mathbb{C}^3$ to be null separated iff the discriminant of this quadratic vanishes, so that the two intersection points $X \cap Y$ coincide.

In the other direction, for a fixed point $[u, \lambda_\alpha] \in \mathbb{MT}$, given one point $x_0 \in \mathbb{C}^3$ obeying (4.11) we can construct the two–parameter family

$$x^{\alpha\beta}(v) = x_0^{\alpha\beta} + v^{(\alpha}\lambda^{\beta)} \tag{4.12}$$

which also obeys (4.11) for any choice of v^α. Thus, a point in $\mathbb{MT}$ corresponds to a totally null complex 2-plane $\mathbb{C}^2 \subset \mathbb{C}^3$.

To consider real Euclidean $\mathbb{R}^3$, rather than $\mathbb{C}^3$, we impose a reality condition on our sections. As in twistor space for four dimensions, consider the antiholomorphic involution $\mathbb{MT} \to \mathbb{MT}$ defined by

$$u \mapsto \hat{u} = \bar{u}\,, \qquad \lambda_\alpha \mapsto \hat{\lambda}_\alpha = (-\bar{\lambda}_1, \bar{\lambda}_0)\,. \tag{4.13}$$

This antiholomorphic involution acts on the $\mathbb{CP}^1$ as the antipodal map, and so has no fixed points. However, there are fixed minitwistor lines. We have

$$\hat{x}^{\alpha\beta} = -\frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} \bar{x} - \mathrm{i}\bar{y} & -\bar{z} \\ -\bar{z} & -\bar{x} - \mathrm{i}\bar{y} \end{pmatrix}, \tag{4.14}$$

so demanding that $\hat{x}^{\alpha\beta} = x^{\alpha\beta}$ imposes $(x, y, z) \in \mathbb{R}^3$ as desired. If $x_0 \in \mathbb{R}^3$ is real Euclidean, then the $\mathbb{C}^2 \subset \mathbb{C}^3$ given by (4.12) intersects the real Euclidean slice only where $v_\alpha = \mathrm{i}r\hat{\lambda}_\alpha$ for some $r \in \mathbb{R}$. Thus, in Euclidean signature, a point in $\mathbb{MT}$ corresponds to a straight line in $\mathbb{R}^3$.

The minitwistor lines of two Euclidean real points still intersect twice in $\mathbb{MT}$; the two intersection points correspond to the two (opposite) orientations of the unique geodesic in $\mathbb{R}^3$ which connects the two points. See Figure 4.1. If one of the intersection points of the two minitwistor lines is (u, λ_α), the norm of the connecting vector $(x-y)^{\alpha\beta}$ is proportional to $\langle\lambda\hat{\lambda}\rangle^2$. This norm is invariant under the Euclidean reality conditions, as required.

So a point in $\mathbb{R}^3$ corresponds to a minitwistor line in $\mathbb{MT}$, while a point in $\mathbb{MT}$ corresponds to an oriented geodesic (i.e., a straight line) in $\mathbb{R}^3$.

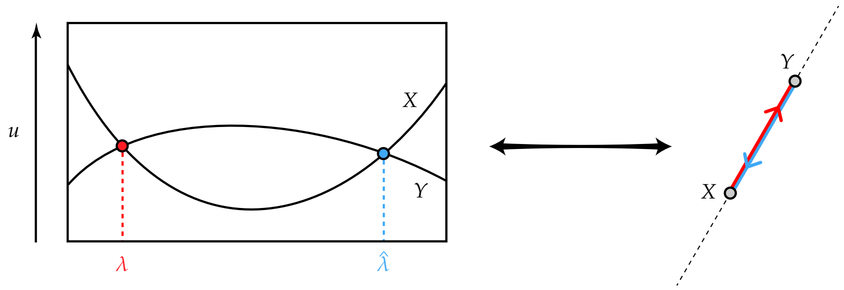

Fig. 4.1 *The correspondence between* $\mathbb{MT}$ *(right) and* $\mathbb{R}^3$ *(left).*

Altogether, the correspondence is summarised by the diagram

$$\begin{tikzcd} & \mathbb{PS} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{MT} & & \mathbb{R}^3 \end{tikzcd}$$

where $\mathbb{PS} \cong \mathbb{R}^3 \times \mathbb{CP}^1$ is the projective spinor bundle with coordinates $(x^{\alpha\beta}, [\lambda_\gamma])$. The projection $\pi_1 : \mathbb{PS} \to \mathbb{MT}$ is given by the incidence relations discussed above, while the second fibration π_2 is the trivial projection $(x, \lambda) \mapsto x$.

For our purposes it will be useful to note that, just as $\mathbb{R}^3$ can be obtained by taking the quotient of $\mathbb{R}^4$ along a constant vector field, so too can $\mathbb{MT}$ be obtained by taking the quotient of the twistor space of $\mathbb{R}^4$ by the action of this vector field on $\mathbb{PT}$ [59, 67, 109]. Explicitly, let $\mathbb{R}^4$ have coordinates $x^{\alpha\dot{\alpha}}$ and let $T = T^{\alpha\dot{\alpha}}\partial/\partial x^{\alpha\dot{\alpha}}$ denote a constant vector field. In the coordinates

$$x^{\alpha\dot{\alpha}} = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} -x+\mathrm{i}y & z+t \\ z-t & x+\mathrm{i}y \end{pmatrix} \qquad \text{we have} \qquad T^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.15}$$

It follows that the three- and four-dimensional coordinates are related as $x^{\alpha\beta} = x^{(\alpha}_{\dot{\alpha}} T^{\beta)\dot{\alpha}}$. The twistor space of $\mathbb{R}^4$ is the total space of the rank-2 bundle $\mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathbb{CP}^1$, and is called $\mathbb{PT}$. In terms of homogeneous coordinates $[Z^A] = [\mu^{\dot{\alpha}}, \lambda_\alpha]$ on $\mathbb{PT}$, the incidence relations are $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$. The vector T defining the symmetry reduction $\mathbb{R}^4 \to \mathbb{R}^3$ acts on $\mathbb{PT}$ as

$$T = T^{\alpha\dot{\alpha}}\lambda_\alpha \frac{\partial}{\partial \mu^{\dot{\alpha}}}. \tag{4.16}$$

Minitwistor space is then obtained by factoring out $\mathbb{PT}$ by the integral curves of this vector field; in particular, the minitwistor coordinate u can be written in terms of coordinates on $\mathbb{PT}$ as $u = \mu_{\dot{\alpha}} T^{\alpha\dot{\alpha}} \lambda_\alpha$, which is annihilated by (4.16) and naturally has scaling weight +2.

The real structure on $\mathbb{MT}$ responsible for Euclidean reality conditions on space–time are inherited from a similar real structure on $\mathbb{PT}$ (c.f., [116]).

We will also be interested in a supersymmetric extension of $\mathbb{MT}$, denoted $\mathbb{MT}_s$. For even $\mathcal{N}$ this is straightforward to achieve: as in [87] we promote $\mathbb{MT}$ to the total space of the bundle

$$\mathcal{O}(2) \oplus \left(\mathbb{C}^{0|\frac{\mathcal{N}}{2}} \otimes \mathcal{O}(1)\right) \to \mathbb{CP}^1\,, \tag{4.17}$$

which can be described by homogeneous coordinates $[u, \chi^a, \lambda_\alpha]$ where the χ^a are Grassmann variables and $a = 1, \ldots, \mathcal{N}/2$. These coordinates are considered up to the rescaling

$$(u,\, \lambda_\alpha,\, \chi^a) \sim (r^2 u,\, r\chi^a,\, r\lambda_\alpha)\,, \tag{4.18}$$

for any $r \in \mathbb{C}^*$. Note that only half the full supersymmetry will be manifest in this description. The incidence relations are similarly generalised to

$$u = x^{\alpha\beta}\, \lambda_\alpha \lambda_\beta\,, \qquad \chi^a = \theta^{a\alpha}\, \lambda_\alpha\,, \tag{4.19}$$

describing a section of (4.17), where $\theta^{a\alpha}$ are coordinates on chiral superspace, written in an SU(4)-invariant formalism. Euclidean reality conditions on the Grassmann variables on $\mathbb{MT}_s$ are inherited from those for supersymmetric extensions of $\mathbb{PT}$ [20]. For example, for $\mathcal{N} = 8$ the conjugation (4.13) is extended to:

$$\chi^a \mapsto \hat{\chi}^a = \left(-\bar{\chi}^2, \bar{\chi}^1, -\bar{\chi}^4, \bar{\chi}^3\right)\,, \tag{4.20}$$

and real minitwistor lines in $\mathbb{MT}_s$ are preserved under this conjugation.

We note that when $\mathcal{N} = 8$ (so $\mathcal{N}/2 = 4$) the Berezinian[2] line bundle has a global holomorphic section given by

$$\Omega_{\mathbb{MT}_s} \equiv \frac{\mathrm{d}u \wedge \mathrm{d}^{2|4}\lambda}{\mathrm{vol}\,\mathbb{C}^*} = \mathrm{d}u \wedge \langle \lambda\, \mathrm{d}\lambda\rangle\, \mathrm{d}^{0|4}\chi\,. \tag{4.21}$$

This has weight zero under the projective rescalings (4.18). Thus $\mathcal{N} = 8$ minitwistor space is a Calabi–Yau supermanifold. We also note that $\Omega_{\mathbb{MT}_s} = T \lrcorner \mathrm{D}^{3|4} Z$ where $\mathrm{D}^{3|4} Z$ is a global holomorphic section of the trivial Berezianian of $\mathcal{N} = 4$ twistor space $\mathbb{PT}_s$.

[2]The Berezinian is the supermanifold analogue of the canonical line bundle.

4.2.2 Penrose transform & Hitchin-Ward correspondence

The Penrose transform represents zero-rest-mass free fields in terms of cohomological data on twistor space [84, 48]. A version of this transform exists for minitwistor space, where it has an interesting interaction with the linearised Bogomolny monopole equations [67, 109, 101]. We now review the relevant aspects of the minitwistor Penrose transform.

Let $f \in \Omega^{0,1}(\mathbb{MT}, \mathcal{O}(-n-2))$ be a (0,1)-form on minitwistor space which is homogeneous of weight $-n-2$. If $n \geq 0$ we can define a spin $n/2$ field on $\mathbb{R}^3$ from this f using the integral transform

$$\varphi_{\alpha_1\cdots\alpha_n}(x) = \int_X \langle\lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\, f|_X\,, \tag{4.22}$$

where $f|_X$ indicates restricting the arguments of f to the minitwistor line $X \subset \mathbb{MT}$ via the incidence relations (4.11). Provided f obeys $\bar{\partial} f = 0$, so that f is holomorphic on $\mathbb{MT}$, it follows that $f|_X = f(x^{\gamma\delta}\lambda_\gamma\lambda_\delta, \lambda_\alpha)$ and the incidence relations ensure that $\varphi_{\alpha_1\cdots\alpha_n}$ obeys the spin $n/2$ free field equations on $\mathbb{R}^3$:

$$\partial^{\alpha_1\beta}\varphi_{\alpha_1\cdots\alpha_n}(x) = \int_X \langle\lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\,\lambda^{\alpha_1}\lambda^{\beta}\,\frac{\partial f}{\partial u}\bigg|_X = 0\,. \tag{4.23}$$

On the other hand, if $f = \bar{\partial} g$ for some $g \in \Omega^0(\mathbb{MT}, \mathcal{O}(-n-2))$ then the spinor field (4.22) is identically zero. Thus, non–trivial solutions to the spin $n/2$ free–field equations correspond to elements of the cohomology class $H^{0,1}(\mathbb{MT}, \mathcal{O}(-n-2))$ and it can be shown that this relation is in fact an isomorphism [67, 101]:

$$H^{0,1}(\mathbb{MT}, \mathcal{O}(-n-2)) \cong \left\{\text{spin } \frac{n}{2} \text{ zero-rest-mass fields on } \mathbb{R}^3\right\}, \tag{4.24}$$

provided the set of zero-rest-mass fields is subject to appropriate analyticity conditions. When $n = 0$ we have

$$f \in H^{0,1}(\mathbb{MT}, \mathcal{O}(-2))\,, \qquad \varphi(x) = \int_X \langle\lambda\, \mathrm{d}\lambda\rangle \wedge f|_X\,, \tag{4.25}$$

corresponding to a (complex) solution of the scalar wave equation $\Box\varphi = 0$ on $\mathbb{R}^3$.

When $n < 0$, the integral transform must be altered. Given a representative $\psi \in H^{0,1}(\mathbb{MT}, \mathcal{O}(-1))$, one can still define a spin $1/2$ zero-rest-mass field by taking

$$\Psi_\alpha(x) = \int_X \langle\lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_\alpha\, \frac{\partial \psi}{\partial u}\bigg|_X\,, \tag{4.26}$$

with $\partial^{\alpha\beta}\Psi_\alpha = 0$ following by the same argument as before. Note that in three dimensions, this spinor has the same Weyl index as that constructed from $f \in H^{0,1}(\mathbb{MT}, \mathcal{O}(-3))$. However, something new occurs for minitwistor fields of weight zero. Given a representative $a \in H^{0,1}(\mathbb{MT}, \mathcal{O})$, we can define both a scalar and a spin-one field on $\mathbb{R}^3$ by the integral transforms

$$\Phi(x) = \int_X \langle \lambda\, \mathrm{d}\lambda\rangle \wedge \left.\frac{\partial a}{\partial u}\right|_X \qquad \text{and} \qquad (*f)_{\alpha\beta}(x) = \int_X \langle \lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_\alpha \lambda_\beta \left.\frac{\partial^2 a}{\partial u^2}\right|_X, \tag{4.27}$$

where we have chosen to write the spin-one field as the dual of a linearised field strength. It is easy to see these fields each obey the relevant field equation $\Box\Phi = 0$ or $\partial^{\alpha\gamma}(*f)_{\alpha\beta} = 0$. However, since they are built from the same twistor field a, one expects that Φ and $*f$ are not independent. In fact, we have

$$\partial_{\alpha\beta}\Phi = \int_X \langle \lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_\alpha \lambda_\beta \left.\frac{\partial^2 a}{\partial u^2}\right|_X = (*f)_{\alpha\beta}, \tag{4.28}$$

showing that a minitwistor representative $a \in H^{0,1}(\mathbb{MT}\mathcal{O})$ encodes both a massless scalar and a Maxwell field which are related by the Bogomolny equations.

The Penrose transform also allows us to encode the full field content of $\mathcal{N} = 8$ SYMH theory into a single field on supersymmetric minitwistor space $\mathbb{MT}_s$. If $\mathcal{A} \in H^{0,1}(\mathbb{MT}_s, \mathcal{O})$, then $\mathcal{A}$ can be expanded in the fermionic directions as

$$\mathcal{A} = a + \chi^a \psi_a + \frac{1}{2}\chi^a\chi^b\varphi_{ab} + \frac{1}{3!}\epsilon_{abcd}\,\chi^a\chi^b\chi^c\tilde{\psi}^d + \frac{\chi^4}{4!}b, \tag{4.29}$$

with the individual $(0,1)$-forms $\{a, \psi_a, \varphi_{ab}, \tilde{\psi}^a, b\}$ on bosonic minitwistor space having weights $0, -1, \ldots, -4$, respectively. Under the Penrose transform, the component fields φ_{ab} correspond to 6 space-time scalars, the fields ψ_a and $\tilde{\psi}^a$ together yield 8 Weyl fermions, whilst a and b together describe both a linearised gluon and a further scalar. As above, the combination that solves the linearised Bogomolny equations is contained in a, whilst b describes a solution to the anti–Bogomolny equations.

This is the dimensional reduction of the statement that on $\mathbb{PT}$, the weight 0 field corresponds to a positive helicity (self–dual) field on $\mathbb{R}^4$, whilst the weight -4 field corresponds to a negative helicity (anti–self–dual) field. Altogether, $\mathcal{A}$ in (4.29) describes the linearised field content of maximal ($\mathcal{N} = 8$) SYMH in three dimensions, in a framework where only an $SU(4)$ subgroup of the full $SO(8)$ R-symmetry is manifest. This subgroup is fixed by choosing which of the seven scalars should be paired with the gauge field in the Bogomolny equations.

At the non-linear level, the *Hitchin–Ward correspondence* [106, 59] describes solutions of the full Bogomolny equations on $\mathbb{R}^3$ in terms of holomorphic vector bundles on minitwistor space. This correspondence is inherited from the Ward construction of instantons in YM_4 via holomorphic vector bundles over the twistor space $\mathbb{PT}$ [105]. The Hitchin–Ward construction is equivalent to other well-known constructions of monopoles [60], such as the Nahm equations [80]. The precise statement of the Hitchin-Ward correspondence is as follows [59]. There is a one-to-one correspondence between $\mathrm{SU}(N)$ Bogomolny monopole configurations on $\mathbb{R}^3$ and rank N holomorphic vector bundles $E \to \mathbb{MT}$ which obey *i)* $E|_X$ is topologically trivial for every minitwistor line $X \subset \mathbb{MT}$, *ii)* $\det E$ is trivial and *iii)* E admits a positive real form. The latter two conditions are related to the choice of special unitary gauge group[3] and the correspondence can be generalised to any choice of gauge group. For our purposes, the most important feature is that holomorphic bundles on $\mathbb{MT}$ correspond to general solutions of the Bogomolny equations on $\mathbb{R}^3$.

4.3 The Minitwistor Action

In this section, we reformulate YMH_3 in terms of minitwistor data. This follows by translating the space-time action (4.6) into a 'minitwistor action.' As we shall see, this action is actually defined on the projective spinor bundle $\mathbb{PS}$, but the equations of motion are naturally phrased in terms of minitwistor space. After formulating the minitwistor action for $\mathcal{N} = 0$ and $\mathcal{N} = 8$ and showing that it has the appropriate equations of motion, we also prove that it reduces to the action on $\mathbb{R}^3$ with a particular choice of gauge.

This construction closely parallels that of the twistor action for Yang-Mills theory in four-dimensions [74, 20, 2], and shares many of its features.

4.3.1 Action and equations of motion

In the introduction to this chapter we saw that YMH_3 admits a perturbative expansion around the Bogomolny monopole sector. This was apparent by writing the action as

$$S[A,B,\Phi] = S_{\mathrm{m}}[A,B,\Phi] + \frac{g^2}{2}\, I[B]\,, \tag{4.30}$$

where

$$S_{\mathrm{m}}[A,B,\Phi] = \int_{\mathbb{R}^3} \mathrm{tr}\,[B \wedge (F - *\mathrm{D}\Phi)] \tag{4.31}$$

[3] Triviality of $\det E$ ensures the existence of a nowhere vanishing holomorphic section of $\det E$ which can be used to normalize the transition matrices of E to have unit determinant. A positive real form on E defines the Killing form on $\mathrm{SU}(N)$.

describes the monopole sector and

$$I[B] = \frac{g^2}{2}\int_{\mathbb{R}^3} \mathrm{tr}\,(B \wedge *B) \tag{4.32}$$

deforms the monopole equations to the full YMH equations. Weak coupling $g = 0$ corresponds to the monopole sector itself.

By the Hitchin–Ward correspondence, the Bogomolny equations $F = *\mathrm{D}\Phi$ are equivalent to holomorphic bundles over $\mathbb{MT}$. Thus, a first attempt at constructing a minitwistor version of S_{m} might be to simply impose these holomorphicity conditions directly on $\mathbb{MT}$ by means of a Lagrange multiplier [87]. Let $E \to \mathbb{MT}$ be a rank N complex (but not necessarily holomorphic) vector bundle, and assume[4] that E is trivial on restriction to the holomorphic lines $X \subset \mathbb{MT}$ corresponding to points $x \in \mathbb{R}^3$. We can endow E with a partial connection $\bar{D}$, locally of the form $\bar{D} = \bar{\partial} + a$ for $a \in \Omega^{0,1}(\mathbb{MT}, \mathcal{O} \otimes \mathrm{End}(E))$. In other words, a is the $(0,1)$-gauge potential for the partial connection $\bar{D}$. If $F^{0,2} = \bar{D}^2$ is the curvature of $\bar{D}$, then a first guess at a minitwistor action for the monopole sector could take the form

$$S_{\mathrm{m}}[a,\beta] \stackrel{?}{=} \int_{\mathbb{MT}} \Omega \wedge \mathrm{tr}\big(\beta\, F^{0,2}\big)\,, \tag{4.33}$$

where $\Omega \equiv \mathrm{d}u \wedge \langle\lambda\,\mathrm{d}\lambda\rangle$ is the top holomorphic form of weight $+4$ on $\mathbb{MT}$ and β is an $\mathrm{End}(E)$-valued function on $\mathbb{MT}$ of weight -4. Varying β, one obtains the equation of motion $F^{0,2} = 0$; by the Hitchin–Ward correspondence, this corresponds to a gauge field A and scalar Φ on $\mathbb{R}^3$ obeying the monopole equations.

Varying a, the other equation of motion we obtain from this action is $\bar{D}\beta = 0$, which imposes that β is globally holomorphic with respect to the complex structure $\bar{D}$. However, the space $H^0(\mathbb{MT}, \mathcal{O}(-4))$ of solutions to this equation is in fact empty, as one can see *e.g.* by restricting β to any minitwistor line, where it would have to be a globally holomorphic function of weight -4 on the Riemann sphere. Thus the second equation implies that $\beta = 0$ and so encodes no physical degrees of freedom on $\mathbb{R}^3$. By contrast, the space-time equations

$$\mathrm{D} * B = 0 \qquad \text{and} \qquad \mathrm{D}B = - * [B,\Phi] \tag{4.34}$$

following from varying (4.31) with respect to Φ *do* have non–trivial solutions, corresponding to linearised anti-Bogomolny fluctuations around the monopole. Thus (4.33) cannot be the minitwistor action corresponding to (4.31).

[4]As in the Hitchin–Ward correspondence, for an $\mathrm{SU}(N)$ theory we must also assume that $\det E$ is trivial and that E has a positive real form.

The remedy for this is perhaps surprising. Rather than constructing an action on $\mathbb{MT}$ itself, we instead consider an action on the projective spinor bundle $\mathbb{PS}$. We now let $E \to \mathbb{PS}$ be a vector bundle over $\mathbb{PS}$ and consider the action[5]

$$S_{\mathrm{m}}[a,b] = \int_{\mathbb{PS}} \Omega \wedge \mathrm{tr}(b \wedge \mathscr{F}) \,, \tag{4.35}$$

where now a and b are $\mathrm{End}(E)$-valued 1-forms on $\mathbb{PS}$ of weight zero and -4, respectively, and $\mathscr{F} = \mathrm{d}a + a \wedge a$ is the curvature of $\mathrm{d} + a$. The action is invariant under the gauge transformations

$$\mathrm{d} + a \to g\,(\mathrm{d} + a)\,g^{-1}\,, \qquad b \to g\,b\,g^{-1}\,, \tag{4.36}$$

for $g \in \Omega^0(\mathbb{PS}, \mathrm{End}(E))$, and also under the shift transformation

$$a \to a\,, \qquad b \to b + (\mathrm{d} + a)\beta\,, \tag{4.37}$$

for β an $\mathrm{End}(E)$-valued function of weight -4 on $\mathbb{PS}$. The latter transformation follows from the Bianchi identity for $\mathscr{F}$, and is standard in BF theories.

A priori, this action may seem a long way from what we are hoping for as both the action and variational data are defined on $\mathbb{PS}$ rather than $\mathbb{MT}$. The equations of motion following from (4.35) state that

$$\Omega \wedge \mathscr{F} = 0 \qquad \text{and} \qquad \Omega \wedge (\mathrm{d}b + [a,b]) = 0\,. \tag{4.38}$$

We can analyse the content of these equations as follows. First note that $\mathbb{PS} \cong \mathbb{MT} \times \mathbb{R}$, so all differential forms on $\mathbb{PS}$ can be expanded in a basis of forms on $\mathbb{MT}$ and the real fibre. It is clear that the action (4.35) only depends on the components of a, b that span the antiholomorphic directions of $\mathbb{PS}$ together with the fibre direction, since the other components wedge to zero against Ω. For the variational problem encoded in this action, one can therefore expand

$$a = \bar{a} + a_\perp\,, \qquad b = \bar{b} + b_\perp\,, \tag{4.39}$$

where $a_\perp$, $b_\perp$ represent 1-forms on $\mathbb{PS}$ pointing along the fibre of the projection $\mathbb{PS} \to \mathbb{MT}$, while $\bar{a}$ and $\bar{b}$ represent forms on $\mathbb{PS}$ that point in the antiholomorphic directions of $\mathbb{MT}$. We can similarly decompose

$$\mathrm{d} = \partial + \bar{\partial} + \mathrm{d}_\perp \tag{4.40}$$

and note that the presence of Ω again means that the exterior derivative ∂ in the holomorphic directions of $\mathbb{MT}$ drops out.

[5]We abuse notation by writing the pullback $\pi_1^*\Omega$ to $\mathbb{PS}$ of the top form on $\mathbb{MT}$ also as Ω.

We now use (4.36) and (4.37) to set $a_\perp = 0 = b_\perp$. As always with axial gauges, this condition does not completely fix the gauge and we are still free to make gauge transformations that are independent of the fibre coordinates – *i.e.* we can still perform gauge transformations on $\mathbb{PS}$ that are the pullback of smooth gauge transforms on $\mathbb{MT}$. In this gauge the equations of motion for $\bar{a}$ read

$$\mathscr{F}^{0,2} = 0 \qquad \text{and} \qquad \mathrm{d}_\perp \bar{a} = 0 \tag{4.41}$$

where $\mathscr{F}^{0,2} = \bar{\partial}\bar{a} + \bar{a}\wedge\bar{a}$, while the equations of motion for $\bar{b}$ become

$$\bar{\partial}\bar{b} + [\bar{a},\bar{b}] = 0 \qquad \text{and} \qquad \mathrm{d}_\perp \bar{b} = 0\,. \tag{4.42}$$

The equations $\mathrm{d}_\perp \bar{a} = 0 = \mathrm{d}_\perp \bar{b}$ tell us that the remaining components $\bar{a}$ and $\bar{b}$ are independent of the real fibre coordinate, so on–shell these $\bar{a}$ and $\bar{b}$ are in fact $(0,1)$-forms on $\mathbb{MT}$, pulled back to $\mathbb{PS}$. The remaining equations $\mathscr{F}^{0,2} = 0$ say that on–shell, the bundle $E \to \mathbb{PS}$ is just the pullback of a holomorphic bundle on $\mathbb{MT}$, while the equation $\bar{D}\bar{b} = 0$ together with the residual gauge invariance says that $\bar{b}$ represents an element of $H^{0,1}(\mathbb{MT},\mathrm{End}(E)\otimes\mathscr{O}(-4))$, pulled back to $\mathbb{PS}$. These are exactly the desired equations on $\mathbb{MT}$, corresponding by the Hitchin–Ward correspondence and the covariant extension of the linear Penrose transform (4.24) to the equations

$$F - *\mathrm{D}\Phi = 0 \qquad \text{and} \qquad \mathrm{D}B = - * [B,\Phi] \tag{4.43}$$

on $\mathbb{R}^3$. Thus, at least on–shell, the action (4.35) corresponds to the action (4.31) on $\mathbb{R}^3$. (We note that both actions vanish trivially when evaluated on a solution of their equations of motion.)

It remains to define the anti-monopole interaction term (4.32) in terms of our fields. In the case of an abelian gauge group, it is clear how to proceed thanks to the Penrose transform. Indeed, at least on–shell we have

$$\int_{\mathbb{R}^3} B\wedge *B = \int\limits_{\mathbb{R}^3\times\mathbb{CP}^1\times\mathbb{CP}^1} \mathrm{d}^3x\,\langle\lambda_1\mathrm{d}\lambda_1\rangle\,\langle\lambda_2\mathrm{d}\lambda_2\rangle\,\langle\lambda_1\lambda_2\rangle^2\,\bar{b}(x,\lambda_1)\,\bar{b}(x,\lambda_2) \tag{4.44}$$

in the abelian case, where $\bar{b}$ is as before.

In the non-abelian case, where b takes values in $\mathrm{End}(E)$, we must modify this term. Since our bundle $E \to \mathbb{PS}$ was assumed to be trivial on restriction to each minitwistor line, on any given minitwistor line there is a smooth gauge transform $h \in \Omega^0(X,\mathrm{End}(E))$ such that

$$h(x,\lambda)\,\bar{D}|_X\,h^{-1}(x,\lambda)=\bar{\partial}|_X\,. \tag{4.45}$$

Clearly, such an h exists thoughout $\mathbb{PS}$ when $a=0$, and so will continue to exist for a sufficiently small. Thus, in perturbation theory, a holomorphic trivialization h will always exist. Furthermore, since $X\subset\mathbb{MT}$ is linearly embedded (via the incidence relations), this holomorphic trivialization is unique up to multiplication $h\mapsto hh_0$ where h_0 is independent of λ. This allows for a *non-abelian* definition of the integral formulae for the Penrose transform (c.f., [74]), with

$$B_{\alpha\beta}(x)=\int_X\langle\lambda\,\mathrm{d}\lambda\rangle\;\lambda_\alpha\lambda_\beta\,h\,\bar{b}|_X\,h^{-1}\,, \tag{4.46}$$

leading to a 1-form on $\mathbb{R}^3$ also valued in the adjoint of the gauge group.

The holomorphic trivialization h can be used to define holomorphic frames [76, 26]

$$U_X(\lambda,\lambda')\equiv h(x,\lambda)\,h^{-1}(x,\lambda')\,, \tag{4.47}$$

which map the fibre of $E|_X$ at λ' to the fibre at λ. By definition,

$$\bar{D}|_X U_X=0\qquad\text{and}\qquad U_X(\lambda,\lambda)=\mathbf{1}\in\mathrm{End}(E)\,. \tag{4.48}$$

The non-abelian generalisation of (4.44) is therefore

$$\begin{aligned}I[a,b]=\int\limits_{\mathbb{R}^3\times\mathbb{CP}^1\times\mathbb{CP}^1}&\mathrm{d}^3x\,\langle\lambda_1\mathrm{d}\lambda_1\rangle\,\langle\lambda_2\mathrm{d}\lambda_2\rangle\,\langle\lambda_1\lambda_2\rangle^2\\&\times\mathrm{tr}\big(b(x,\lambda_1)\,U_X(\lambda_1,\lambda_2)\,b(x,\lambda_2)\,U_X(\lambda_2,\lambda_1)\big)\,,\end{aligned} \tag{4.49}$$

with the holomorphic frames serving to transport the insertions of b between the two different insertion points on $X\cong\mathbb{CP}^1$.

The full minitwistor action is

$$S[a,b]=S_m[a,b]+\frac{g^2}{2}\,I[a,b]\,. \tag{4.50}$$

Note that the anti–monopole interaction term $I[a,b]$ is independent of the components $a_\perp$ and $b_\perp$ pointing along the fibres of $\mathbb{PS}\to\mathbb{MT}$, since these components wedge to zero in the perturbative expansion of (4.49). So on–shell, it remains true that $\bar{a}$ and $\bar{b}$ are independent of the fibre directions, and they can be considered to be pulled back from forms on $\mathbb{MT}$. However, the equations of motion no longer imply that the bundle over $\mathbb{MT}$

is holomorphic. In fact, one finds

$$\mathscr{F}^{0,2} = g^2\, \mathrm{d}^2 x_{(0,2)} \int_X \langle \lambda' \mathrm{d}\lambda' \rangle \langle \lambda\, \lambda' \rangle^2\, U_X(\lambda, \lambda')\, \bar{b}(x, \lambda')\, U_X(\lambda', \lambda)\,, \tag{4.51}$$

and

$$\begin{aligned}\bar{D} b = g^2 \frac{\mathrm{d}^2 x_{(0,2)}}{\langle \lambda\, \hat{\lambda} \rangle^2} \int\limits_{\mathbb{CP}^1 \times \mathbb{CP}^1} & \langle \lambda' \mathrm{d}\lambda' \rangle \langle \lambda'' \mathrm{d}\lambda'' \rangle \langle \lambda' \hat{\lambda} \rangle \langle \lambda'' \hat{\lambda} \rangle \langle \lambda' \lambda'' \rangle \\ & \times \left[U_X(\lambda, \lambda')\, \bar{b}(x, \lambda')\, U_X(\lambda', \lambda),\, U_X(\lambda, \lambda'')\, \bar{b}(x, \lambda'')\, U_X(\lambda'', \lambda) \right], \end{aligned} \tag{4.52}$$

where the 2-forms

$$\mathrm{d}x^{\alpha\beta}_{(0,2)} := \mathrm{d}x^{\delta\gamma} \wedge \mathrm{d}x^{\sigma}_{\gamma} \frac{\lambda^\alpha \lambda^\beta\, \hat{\lambda}_\delta \hat{\lambda}_\sigma}{\langle \lambda\, \hat{\lambda} \rangle^2}\,, \qquad \mathrm{d}^2 x_{(0,2)} := \frac{\hat{\lambda}_\alpha \hat{\lambda}_\beta}{\langle \lambda\, \hat{\lambda} \rangle^2}\, \mathrm{d}^2 x^{\alpha\beta}_{(0,2)}\,,$$

are projected to point in the anti-holomorphic directions of $\mathbb{MT}$. The two equations (4.51) and (4.52) are in fact equivalent to the field equations of YMH theory, written in the form (4.7); this follows from an argument similar to the one given for the four-dimensional Yang-Mills equations in [74]. We will not show this in detail here, because in the next section we obtain the stronger result that the actions (4.50) and (4.30) are in fact equivalent *off–shell*.

Generalising the minitwistor action to $\mathcal{N} = 8$ SYMH$_3$ theory is straightforward. Encoding the field content into a single multiplet $\mathscr{A}$ (defined initially on $\mathbb{PS}$), the resulting action takes the form:

$$S[\mathscr{A}] = \int_{\mathbb{PS}_s} \Omega_s \wedge \mathrm{tr}\left(\mathscr{A} \wedge \mathrm{d}\mathscr{A} + \frac{2}{3} \mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A} \right) + \frac{g^2}{2} \int_{\mathbb{R}^{3|8}} \mathrm{d}^{3|8} x\, \log\det\left(\bar{\partial} + \mathscr{A}\right)|_X\,. \tag{4.53}$$

The first term here is a ‘partially holomorphic’ Chern-Simons theory on $\mathbb{PS}$, which was shown to describe the supersymmetric Bogomolny sector in [87]. The $\log\det(\bar{\partial} + \mathscr{A})|_X$ term can be understood perturbatively, via the expansion

$$\log\det\left(\bar{\partial} + \mathscr{A}\right)|_X = \mathrm{tr}\left(\log \bar{\partial}|_X\right) + \sum_{n=2}^{\infty} \frac{1}{n} \int_{(\mathbb{CP}^1)^{\times n}} \mathrm{tr}\left(\bar{\partial}^{-1}|_X \mathscr{A}(\lambda_1) \cdots \bar{\partial}^{-1}|_X \mathscr{A}(\lambda_n) \right), \tag{4.54}$$

where $\bar\partial|_X$ is the $\bar\partial$-operator along a minitwistor line. We can also write

$$\int_{(\mathbb{CP}^1)^{\times n}} \operatorname{tr}\big(\bar\partial^{-1}|_X\mathscr{A}(\lambda_1)\cdots\bar\partial^{-1}|_X\mathscr{A}(\lambda_n)\big) = \int_{(\mathbb{CP}^1)^{\times n}} \frac{\langle\lambda_1 \mathrm{d}\lambda_1\rangle\cdots\langle\lambda_n\mathrm{d}\lambda_n\rangle}{\langle\lambda_1\lambda_2\rangle\langle\lambda_2\lambda_3\rangle\cdots\langle\lambda_n\lambda_1\rangle}\operatorname{tr}\left(\mathscr{A}(\lambda_1)\mathscr{A}(\lambda_2)\cdots\mathscr{A}(\lambda_n)\right), \quad (4.55)$$

using the Cauchy kernel on $\mathbb{CP}^1$.

The action (4.50) is obviously very closely related to the twistor action

$$S[\mathscr{A}_4] = \int_{\mathbb{PT}_s} \mathrm{D}^{3|4}Z\wedge\operatorname{tr}\left(\mathscr{A}_4\wedge\bar\partial\mathscr{A}_4+\frac{2}{3}\mathscr{A}_4\wedge\mathscr{A}_4\wedge\mathscr{A}_4\right)+\frac{g_4^2}{2}\int_{\mathbb{R}^{4|8}}\mathrm{d}^{4|8}x\,\log\det\big(\bar\partial+\mathscr{A}_4\big)|_X \quad (4.56)$$

obtained in [20] that describes $\mathcal{N}=4$ SYM in four dimensions. The first term in (4.56) is an integral over the $\mathcal{N}=4$ twistor space $\mathbb{PT}_s\cong\mathscr{O}(1)\otimes\mathbb{C}^{2|4}\to\mathbb{CP}^1$, where $\mathscr{A}_4\in\Omega^{0,1}(\mathbb{PT}_s,\mathrm{End}(E))$, while the second term is an integral over the space of real Euclidean twistor lines $X\subset\mathbb{PT}_s$, corresponding to chiral $\mathcal{N}=4$ superspace in four dimensions. Indeed, if $\mathscr{L}_T(\mathscr{A}_4)=0$ so that $\mathscr{A}_4$ is invariant along the flow of the vector field $T=T^{\alpha\dot\alpha}\lambda_\alpha\partial/\partial\mu^{\dot\alpha}$ that reduces $\mathbb{PT}_s$ to $\mathbb{MT}_s$, then identifying

$$\Omega_s = T\lrcorner\mathrm{D}^{3|4}Z \qquad \text{and} \qquad \mathrm{d}^{3|8}x = T\lrcorner\mathrm{d}^{4|8}x, \quad (4.57)$$

we obtain the minitwistor action (4.53) upon taking the symmetry reduction from $\mathbb{PT}_s$ along the integrable curves of T.

4.3.2 Equivalence to space-time action

We now establish the off–shell equivalence of the action (4.53) with the $\mathcal{N}=8$ SYMH action on $\mathbb{R}^3$. This of course also follows from its status as the symmetry reduction of the twistor action (4.56) for $\mathcal{N}=4$ SYM, but we prefer to show it here from scratch. We first choose a basis of forms on $\mathbb{PS}$ that is adapted to the description $\mathbb{PS}\cong\mathbb{R}^3\times\mathbb{CP}^1$. Define the forms

$$\bar e^0 \equiv \frac{\langle\hat\lambda\,\mathrm{d}\hat\lambda\rangle}{\langle\lambda\,\hat\lambda\rangle^2} \qquad \text{and} \qquad \bar e^\alpha \equiv \frac{\mathrm{d}x^{\alpha\beta}\,\hat\lambda_\beta}{\langle\lambda\,\hat\lambda\rangle} \quad (4.58)$$

which are dual to the vector fields

$$\bar\partial_0 \equiv \langle\lambda\,\hat\lambda\rangle\,\lambda_\alpha\frac{\partial}{\partial\hat\lambda_\alpha} \qquad \text{and} \qquad \bar\partial_\alpha \equiv \lambda^\beta\frac{\partial}{\partial x^{\alpha\beta}}, \quad (4.59)$$

respectively. We have chosen to normalize these forms and vector fields by powers of $\langle\lambda\,\hat\lambda\rangle$ such that they have no antiholomorphic weight. Note also that

$$\Omega\wedge\bar e^0\wedge\bar e^\alpha\wedge\bar e_\alpha=\frac{\langle\lambda\,\mathrm{d}\lambda\rangle\wedge\langle\hat\lambda\,\mathrm{d}\hat\lambda\rangle}{\langle\lambda\,\hat\lambda\rangle^2}\wedge\mathrm{d}^3x\,, \tag{4.60}$$

so (4.58) span the directions of $\mathbb{PS}$ not involved in the holomorphic form Ω.

In terms of the basis (4.58) we can expand the field $\mathscr{A}$ as

$$\mathscr{A}(x,\lambda,\hat\lambda,\theta\lambda)=\mathscr{A}_0(x,\lambda,\hat\lambda,\theta\lambda)\,\bar e^0+\mathscr{A}_\alpha(x,\lambda,\hat\lambda,\theta\lambda)\,\bar e^\alpha\,. \tag{4.61}$$

We now exploit the gauge redundancy of the action (4.53) to impose the 'spacetime gauge'

$$\bar\partial^*|_X\mathscr{A}|_X=0 \tag{4.62}$$

on every minitwistor line X, where $\bar\partial^*|_X$ is the adjoint of $\bar\partial|_X$ with respect to the standard Fubini–Study metric on $X\cong\mathbb{CP}^1$. (The action does not require any choice of metric except through this gauge–fixing term.) We also have $(\bar\partial\mathscr{A})|_X=0$ for trivial dimensional reasons, so in this gauge $\mathscr{A}|_X$ is fixed to be a harmonic representative of the cohomology group

$$H^{0,1}(X,\mathrm{End}(E)\otimes\mathscr{O}_s)\cong\bigoplus_{n=0}^{4}H^{0,1}(X,\mathrm{End}(E)\otimes\mathscr{O}_X(-n))\otimes\mathbb{C}^{\frac{4!}{n!(4-n)!}}\,, \tag{4.63}$$

where the right hand side gives the cohomology groups describing the component fields in the expansion of the supermultiplet $\mathscr{A}=a+\cdots+\chi^4 b$. These cohomology groups vanish if $n<2$ so the component fields $a|_X$ and $\psi_a|_X$ vanish. Harmonic representatives for the remaining fields are [116]

$$\varphi_{ab}|_X=\phi_{ab}(x)\,\bar e^0\,,\qquad\tilde\psi^a|_X=2\,\frac{\tilde\Psi^a_\alpha(x)\,\hat\lambda^\alpha}{\langle\lambda\,\hat\lambda\rangle}\,\bar e^0\qquad\text{and}\qquad b|_X=3\,\frac{B_{\alpha\beta}(x)\,\hat\lambda^\alpha\hat\lambda^\beta}{\langle\lambda\,\hat\lambda\rangle^2}\,\bar e^0 \tag{4.64}$$

where the fields $\{\phi_{ab},\tilde\Psi^a_\alpha,B_{\alpha\beta}\}$ can depend only on $x\in\mathbb{R}^3$. While the components of $\mathscr{A}$ restricted to (real Euclidean) minitwistor lines are fixed, our gauge condition does not constrain the remaining components of $\mathscr{A}$. Thus, in this gauge we have

$$\begin{aligned}
a&=a_\alpha(x,\lambda,\hat\lambda)\,e^\alpha\,, & \psi_a&=\psi_{a\alpha}(x,\lambda,\hat\lambda)\,\bar e^\alpha\,,\\
\varphi_{ab}&=\phi_{ab}(x)\,\bar e^0+\varphi_{ab\,\alpha}(x,\lambda,\hat\lambda)\,\bar e^\alpha\,, & \tilde\psi^a&=2\,\frac{\tilde\Psi^a_\alpha(x)\,\hat\lambda^\alpha}{\langle\lambda\,\hat\lambda\rangle}\,\bar e^0+\tilde\psi^a_\alpha(x,\lambda,\hat\lambda)\,\bar e^\alpha\,,\\
b&=3\,\frac{B_{\alpha\beta}(x)\,\hat\lambda^\alpha\hat\lambda^\beta}{\langle\lambda\,\hat\lambda\rangle^2}\,\bar e^0+b_\alpha(x,\lambda,\hat\lambda)\,\bar e^\alpha\,.
\end{aligned} \tag{4.65}$$

This gauge is not a complete gauge fixing on $\mathbb{PS}$, with the residual gauge freedom being smooth gauge transformations γ which obey

$$\bar{\partial}^*|_X \bar{\partial}_X \gamma(x,\lambda,\hat{\lambda}) = 0 \tag{4.66}$$

and so are themselves harmonic on minitwistor lines. Since γ is homogeneous of weight zero on $\mathbb{CP}^1$, by the maximum modulus principle it follows that such $\gamma(x,\lambda,\hat{\lambda}) = \gamma(x)$. Thus, in harmonic gauge on $\mathbb{PS}$, the residual gauge freedom of the minitwistor action is just ordinary gauge transformations on $\mathbb{R}^3$.

We now evaluate the minitwistor action using the harmonic gauge fields (4.65). Consider first the monopole contribution $S_{\rm m}[\mathscr{A}]$. Performing the Grassmann integration over $\mathrm{d}^4\chi$ is straightforward, leaving

$$\begin{aligned}\int_{\mathbb{PS}} \mathrm{d}^3x \wedge \omega \, \mathrm{tr}\Bigg[& 3\,\frac{B_{\alpha\beta}(x)\,\hat{\lambda}^\alpha\hat{\lambda}^\beta}{\langle\lambda\,\hat{\lambda}\rangle^2}\left(\lambda^\delta\partial^\gamma_\delta a_\gamma + \frac{1}{2}[a^\gamma, a_\gamma]\right) + \phi^{ab}\,\psi_{a\alpha}\,\psi^\alpha_b \\ & + 2\,\frac{\widetilde{\Psi}^a_\alpha(x)\,\hat{\lambda}^\alpha}{\langle\lambda\,\hat{\lambda}\rangle}\left(\lambda^\beta\partial^\gamma_\beta\psi_{a\gamma} + [a^\gamma,\psi_{a\gamma}]\right) + \frac{\phi^{ab}}{2}\left(\lambda^\alpha\partial^\beta_\alpha\varphi_{ab\beta} + [a^\beta,\varphi_{ab\beta}]\right) \\ & - b^\alpha\bar{\partial}_0 a_\alpha - \tilde{\psi}^{a\alpha}\bar{\partial}_0\psi_{a\alpha} + \frac{1}{2}\varphi^{ab}_\alpha\,\bar{\partial}_0\varphi^\alpha_{ab}\Bigg]\,,\end{aligned} \tag{4.67}$$

where

$$\omega \equiv \frac{\langle\lambda\,\mathrm{d}\lambda\rangle\wedge\langle\hat{\lambda}\,\mathrm{d}\hat{\lambda}\rangle}{\langle\lambda\,\hat{\lambda}\rangle^2}\,, \tag{4.68}$$

is the Kähler form on $\mathbb{CP}^1$. The field components b_α and $\tilde{\psi}^a_\alpha$ appear only in the third line of (4.67). Integrating them out of the path integral enforces the constraints

$$\bar{\partial}_0 a_\alpha = 0\,, \qquad \bar{\partial}_0\psi_{a\alpha} = 0\,, \tag{4.69}$$

so that a_α and $\psi_{a\alpha}$ must be globally holomorphic in λ. Accounting for the weight -1 of the basis form $\bar{e}^\alpha$, we see that $\psi_{a\alpha}$ is homogeneous of weight zero with respect to λ (and $\hat{\lambda}$), whilst a_α is homogeneous of weight $+1$ in λ. Thus the second of these constraints implies that $\psi_{a\alpha} = \Psi_{a\alpha}(x)$, whilst the first constraint implies that $a_\alpha = \lambda^\beta\tilde{A}_{\alpha\beta}(x)$. Decomposing this $\tilde{A}_{\alpha\beta}$ into its symmetric and anti-symmetric parts, the result is

$$a_\alpha(x,\lambda,\hat{\lambda}) = \lambda^\beta A_{\alpha\beta}(x) - 2\lambda_\alpha\,\Phi(x) \qquad \text{and} \qquad \psi_{a\alpha}(x,\lambda,\hat{\lambda}) = \Psi_{a\alpha}(x)\,, \tag{4.70}$$

where $A_{\alpha\beta} = A_{\beta\alpha}$ defines a gauge field on $\mathbb{R}^3$ and $\Phi(x)$ is the Higgs field.

Having solved these constraints, we can perform the path integral over the components $\varphi_{ab\,\alpha}$, which is a Gaussian with quadratic operator $\bar\partial_0$. The result of this path integration leaves an action

$$\int\limits_{\mathbb{R}^3\times\mathbb{CP}^1} \mathrm{d}^3x\wedge\omega\,\mathrm{tr}\Bigg[3\,\frac{B_{\alpha\beta}\,\hat\lambda^\alpha\,\hat\lambda^\beta}{\langle\lambda\,\hat\lambda\rangle^2}\lambda^\gamma\lambda^\delta\left(\partial^\kappa_\gamma A_{\kappa\delta}+[A^\kappa_\gamma,\,A_{\kappa\delta}]-2\mathrm{D}_{\gamma\delta}\Phi\right)+\phi^{ab}\,\Psi_{a\,\alpha}\,\Psi^\alpha_b$$
$$+2\,\frac{\widetilde\Psi^a_\alpha\,\hat\lambda^\alpha\lambda^\beta}{\langle\lambda\,\hat\lambda\rangle}\left(\mathrm{D}^\gamma_\beta\Psi_{a\gamma}+[\Phi,\,\Psi_{a\beta}]\right)+\frac{1}{4}\,[\phi^{ab},\,\Phi]\,[\Phi,\,\phi_{ab}]$$
$$+\frac{\phi^{ab}\,\lambda^\alpha\,\hat\lambda^\beta}{2\,\langle\lambda\,\hat\lambda\rangle}\left(\mathrm{D}^\gamma_\alpha\mathrm{D}_{\beta\gamma}\phi_{ab}+\partial_{\alpha\beta}[\Phi,\,\phi_{ab}]\right)\Bigg]\,. \tag{4.71}$$

At this point, the integral over the $\mathbb{CP}^1$ factor of $\mathbb{PS}\cong\mathbb{CP}^1\times\mathbb{R}^3$ can be performed using the rule [20]

$$\int_{\mathbb{CP}^1}\frac{\omega}{\langle\lambda\,\hat\lambda\rangle^n}\lambda^{\alpha_1}\cdots\lambda^{\alpha_n}S_{\alpha_1\cdots\alpha_n}\,\hat\lambda_{\beta_1}\cdots\hat\lambda_{\beta_n}\,T_{\beta_1\cdots\beta_n}=\frac{1}{n+1}S_{\alpha_1\cdots\alpha_n}\,T^{\alpha_1\cdots\alpha_n}\,, \tag{4.72}$$

which is a consequence of Serre duality on the Riemann sphere. The result is

$$\int_{\mathbb{R}^3}\mathrm{d}^3x\,\mathrm{tr}\Big\{B_{\alpha\beta}\left(\partial^{\alpha\gamma}A^\beta_\gamma+[A^{\alpha\gamma},\,A^\beta_\gamma]-2\mathrm{D}^{\alpha\beta}\Phi\right)-\widetilde\Psi^a_\alpha\,\mathrm{D}^{\alpha\beta}\Psi_{a\beta}+\Psi_{a\alpha}\,\Phi\,\widetilde\Psi^{a\alpha}$$
$$+\Psi_{a\alpha}\Psi^\alpha_b\,\phi^{ab}+\frac{1}{4}\phi_{ab}\,\mathrm{D}_{\alpha\beta}\mathrm{D}^{\alpha\beta}\phi^{ab}+\frac{1}{4}[\phi_{ab},\Phi]\,[\Phi,\phi^{ab}]\Big\}\,. \tag{4.73}$$

This is equal to the monopole action for $\mathcal{N}=8$ SYMH theory on $\mathbb{R}^3$, up to a total derivatives which can be discarded.

The calculation for the interaction term $I[\mathscr{A}]$ follows similar lines. In harmonic gauge, the perturbative expansion of $\log\det(\bar\partial+\mathscr{A})|_X$ terminates at fourth-order (because $\mathscr{A}_0$ goes like χ^2 at leading order in this gauge), so there are relatively few terms to consider. These are further reduced by the requirement that the fermionic integral over $\mathrm{d}^8\theta$ must be saturated. As it turns out, only three such terms are present in the perturbative expansion: one at second order ($\sim b^2$), one at third order ($\sim\phi\widetilde\psi\widetilde\psi$), and one at fourth order ($\sim\phi^4$). We will only review the calculation for the second order term; the others follow a similar path.

The relevant second-order contribution is

$$\int \mathrm{d}^{3|8}x \int_{(\mathbb{CP}^1)^{\times 2}} \frac{\langle\lambda_1\, \mathrm{d}\lambda_1\rangle}{\langle\lambda_2\,\lambda_1\rangle} b_0(x,\lambda_1,\hat{\lambda}_1)\,\bar{e}_1^0\,\chi_1^4 \frac{\langle\lambda_2\,\mathrm{d}\lambda_2\rangle}{\langle\lambda_1\,\lambda_2\rangle} b_0(x,\lambda_2,\hat{\lambda}_2)\,\bar{e}_2^0\,\chi_2^4$$

$$= -9\int \mathrm{d}^{3|8}x \int_{(\mathbb{CP}^1)^{\times 2}} \frac{\omega_1\,\omega_2}{\langle\lambda_1\,\lambda_2\rangle^2} \frac{B_{\alpha\beta}\,\hat{\lambda}_1^{\alpha}\,\hat{\lambda}_1^{\beta}}{\langle\lambda_1\,\hat{\lambda}_1\rangle^2} (\theta^{a\gamma}\lambda_{1\gamma})^4 \frac{B_{\delta\kappa}\,\hat{\lambda}_2^{\delta}\,\hat{\lambda}_2^{\kappa}}{\langle\lambda_2\,\hat{\lambda}_2\rangle^2} (\theta^{b\sigma}\lambda_{2\sigma})^4, \quad (4.74)$$

where the incidence relations $\chi^a = \theta^{a\alpha}\lambda_\alpha$ have been used. It can be shown that

$$\int \mathrm{d}^8\theta\,(\theta^{a\gamma}\lambda_{1\gamma})^4\,(\theta^{b\beta}\lambda_{2\beta})^4 = \langle\lambda_1\lambda_2\rangle^4, \quad (4.75)$$

which enables the further reduction of the second-order contribution to:

$$-9 \int_{\mathbb{R}^3\times\mathbb{CP}^1\times\mathbb{CP}^1} \mathrm{d}^3x \frac{\omega_1\,\omega_2}{\langle\lambda_1\,\hat{\lambda}_1\rangle^2\,\langle\lambda_2\,\hat{\lambda}_2\rangle^2} \langle\lambda_1\,\lambda_2\rangle\, B_{\alpha\beta}\,\hat{\lambda}_1^{\alpha}\,\hat{\lambda}_1^{\beta}\, B_{\gamma\delta}\hat{\lambda}_2^{\gamma}\,\hat{\lambda}_2^{\delta}$$

$$= -\int_{\mathbb{R}^3} \mathrm{d}^3x\, B_{\alpha\beta}\, B^{\alpha\beta}, \quad (4.76)$$

after making use of (4.72). This is precisely the B^2 contribution to the anti-monopole interactions for $\mathcal{N} = 8$ SYMH$_3$. The other two terms in the space-time action are generated by the third and fourth order contributions from the perturbative expansion of log det in a similar fashion.

4.3.3 Other Reality Conditions

In three dimensions, it is possible to impose reality conditions which result in the more physical 2,1 signature [108, 104, 46]. Writing $\lambda_\alpha = (1, z)$, and the spacetime point $x^{\alpha\beta}$ as

$$x^{\alpha\beta} = \begin{pmatrix} t+x & y \\ y & t-x \end{pmatrix}, \quad (4.77)$$

where each of t, x and y are real in order to obtain Lorentzian 3-dimensional space, the incidence relation becomes

$$u = (t+x)z^2 + 2yz + (t-x). \quad (4.78)$$

If z (and hence u) are both real, this is a 2-dimensional spacetime null plane with normal

$$n = \begin{pmatrix} z^2+1 \\ z^2-1, \\ 2z \end{pmatrix} \tag{4.79}$$

and this is a null vector.

However, such reality conditions leave the space $\mathbb{PS}$ with only four real degrees of freedom: the three components of $x^{\alpha\beta}$ and z. The Bogomolny part of the action on $\mathbb{PS}$ is an integral of the Chern-Simons 3-form against the measure $\langle\lambda d\lambda\rangle \wedge du$, which is altogether a 5-form. This is too large to be integrated over a four dimensional space so a different sort of action would be required for Lorenztian signature.

4.4 Tree Amplitudes in YMH$_3$ Theory

For $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions, the perturbative expansion around the self–dual sector has its ultimate expression in the RSVW formula [115, 89]

$$\sum_{d=0}^{\infty} g_{(4)}^{2d} \int \mathrm{d}\tilde{\mu}_{\tilde{C}}^{(d)} \log\det(\bar{\partial}+\mathscr{A}_4)|_{\tilde{C}} \tag{4.80}$$

where $\tilde{C}$ is a the image of a degree d holomorphic map $Z:\mathbb{CP}^1 \to \mathbb{CP}^{3|4}$ from a rational curve to $\mathcal{N}=4$ twistor space $\mathbb{CP}^{3|4}$, while

$$\mathrm{d}\tilde{\mu}_{\tilde{C}}^{(d)} = \frac{\mathrm{d}^{4|4\times(d+1)}Z}{\mathrm{vol}\,\mathrm{GL}(2;\mathbb{C})}, \tag{4.81}$$

is a top holomorphic form on the moduli space of all such maps, described in terms of homogeneous coordinates on the target and considered up to automorphisms of the source curve[6], and $\bar{\partial}+\mathscr{A}_4$ is a (0,1)–connection on a complex holomorphic bundle $E \to \mathbb{CP}^{3|4}$. Expanding in powers of the on-shell background field $\mathscr{A}_4$, this formula is a generating functional for all tree amplitudes in $\mathcal{N}=4$ SYM$_4$. The degree of the map indicates the grading of the scattering amplitude by NMHV degree, with a degree d map corresponding to a N^{d-1}MHV tree amplitude.

Since Yang-Mills-Higgs theory inherits its perturbative expansion around solutions of the Bogomolny equations from the MHV expansion of Yang-Mills theory in four di-

[6] It is easily checked that, like $\mathbb{CP}^{3|4}$ itself, the moduli space is a Calabi–Yau supermanifold. See also [115, 77, 6]

mensions, it is natural to ask if a similar connected prescription exists for the tree-level S-matrix of YMH theory in three dimensions. Indeed, a formula along these lines was given in [28] as a literal restriction of the RSVW formula to three–dimensional kinematics. In this section we present a new formula that is adapted to the minitwistor geometry appropriate for the three–dimensional theory.

4.4.1 A connected prescription generating functional

In three dimensions an on-shell gluon has only one polarization state, so we cannot hope to have any analogue of an 'MHV' expansion for pure Yang-Mills theory. However, in YMH$_3$ theory it is natural to grade n-particle perturbative amplitudes according to how many of the external particles depart from solutions of the (linearised) Bogomlony equations.

On $\mathbb{MT}_s$, the amplitudes of $\mathcal{N}=8$ SYMH$_3$ theory can be viewed as functionals of the on-shell supermultiplet $\mathscr{A}$, given by (4.29). As in four dimensions, the fermionic expansion of this supermultiplet automatically keeps track of this three–dimensional 'MHV' expansion, as noted in [35, 10]. In particular, any tree-level amplitude $\mathscr{M}_n^{(0)}$ can be expanded as a polynomial in the fermionic components of the on-shell supermomenta $\{\chi_i^a\}$, starting with a term of order 4 and truncating at order $4(n-2)$. The k^{th} term in this expansion is of order $4(k+2)$ and is identified as the N^{k}MHV superamplitude; if we project out all but the top and bottom components of the supermultiplet, this N^{k}MHV amplitude would contain $k+2$ external states that obey the linearised 'anti–Bogomolny' equations.

With this understanding, our formula is remarkably similar to (4.80): we find that all amplitudes in $\mathcal{N}=8$ SYMH theory in three dimensions are given by the generating functional

$$\sum_{d=0}^{\infty} g^{2d} \int \mathrm{d}\mu_C^{(d)} \, \frac{1}{R(\lambda)} \log\det(\bar\partial + \mathscr{A})|_C \,. \tag{4.82}$$

As in the RSVW formula, $\bar\partial + \mathscr{A}$ is a (0,1)-connection on a background complex holomorphic bundle, here over $\mathbb{MT}_s$. C is the image of a degree d map

$$\begin{aligned} Z : \mathbb{CP}^1 &\to \mathbb{MT}_s \,, \\ : [\sigma^{\mathbf{a}}] &\mapsto \big[u(\sigma), \lambda_\alpha(\sigma), \chi^a(\sigma)\big] \end{aligned} \tag{4.83}$$

from a Riemann sphere, described by homogeneous coordinates $[\sigma^{\mathbf{a}}] = [\sigma^{\mathbf{0}}, \sigma^{\mathbf{1}}]$, to maximally supersymmetric minitwistor space. Explicitly, we have

$$\begin{gathered} u(\sigma) = u_{\mathbf{a}_1\cdots\mathbf{a}_{2d}} \sigma^{\mathbf{a}_1} \cdots \sigma^{\mathbf{a}_{2d}} \,, \qquad \lambda_\alpha(\sigma) = \lambda_\alpha^{\mathbf{a}_1\cdots\mathbf{a}_d} \sigma_{\mathbf{a}_1} \cdots \sigma_{\mathbf{a}_d} \,, \\ \chi^a(\sigma) = \chi^a_{\mathbf{b}_1\cdots\mathbf{b}_d} \sigma^{\mathbf{b}_1} \cdots \sigma^{\mathbf{b}_d} \,. \end{gathered} \tag{4.84}$$

Note that the polynomial $u(\sigma)$ has degree $2d$, since it scales with twice the weight of λ_α and χ^a on minitwistor space.

The measure

$$\mathrm{d}\mu_C^{(d)} = \frac{\mathrm{d}^{2d+1}u\,\mathrm{d}^{2|4\times(d+1)}\lambda}{\mathrm{vol\,GL}(2;\mathbb{C})} \tag{4.85}$$

is a top holomorphic form on the space of such maps. Note that with this measure, the moduli space is not canonically Calabi-Yau: under a rescaling

$$(u,\lambda_\alpha,\chi^a) \to (r^2u, r\lambda_\alpha, r\chi^a) \tag{4.86}$$

of the homogeneous coordinates on $\mathbb{MT}_s$, one finds

$$\mathrm{d}\mu_C^{(d)} \to r^{2(2d+1)}r^{(2-4)(d+1)}\,\mathrm{d}\mu_C^{(d)} = r^{2d}\,\mathrm{d}\mu_C^{(d)} \tag{4.87}$$

so that $\mathrm{d}\mu_C^{(d)}$ has non–trivial scaling weight. This weight is compensated by the new ingredient $R(\lambda)$ which is the *resultant* of the λ components of the map $\lambda:\mathbb{CP}^1\to\mathbb{CP}^1$ (*c.f.*, [27]). By definition, this resultant is a homogeneous polynomial in the coefficients $\lambda_\alpha^{\mathbf{a}_1\cdots\mathbf{a}_d}$ of degree d that vanishes if and only if the two polynomials $\lambda_\alpha(\sigma)$ have a simultaneous root; in other words, if there is some point $[\sigma_*]\in\mathbb{CP}^1$ for which $\lambda_\alpha(\sigma_*)=0$. Since λ describes a map to $\mathbb{CP}^1$, this never occurs, so $1/R(\lambda)$ is nowhere singular. Since it is homogeneous of degree d, the resultant scales under (4.86) as

$$R(r\lambda)\to r^{2d}\,R(\lambda)\,. \tag{4.88}$$

so that the measure $\mathrm{d}\mu_C^{(d)}/R(\lambda)$ is in fact scale invariant and holomorphic.

Clearly, the formula (4.82) shares many features with its four-dimensional avatar, the RSVW formula. Indeed, as we will show below, it can be seen as a straightforward symmetry reduction of the RSVW formula.

We now demonstrate that (4.82) does indeed yield the correct tree amplitudes. Expanding in powers of the background field $\mathscr{A}$, the n-particle tree-level N^{d-1}MHV superamplitude is given by

$$\mathscr{M}_{n,d}^{(0)} = \int \frac{\mathrm{d}\mu_C^{(d)}}{R(\lambda)} \prod_{i=1}^{n}\frac{(\sigma_i\mathrm{d}\sigma_i)}{(i\,i{+}1)}\,\mathrm{tr}\,(\mathscr{A}_1\mathscr{A}_2\cdots\mathscr{A}_n)\,, \tag{4.89}$$

where $\mathscr{A}_i=\mathscr{A}(u(\sigma_i),\lambda(\sigma_i),\chi(\sigma_i))$, $(\sigma_i\mathrm{d}\sigma_i)=\epsilon_{\mathbf{ab}}\,\sigma_i^{\mathbf{a}}\mathrm{d}\sigma_i^{\mathbf{b}}$ and similarly $(i\,i{+}1)=\epsilon_{\mathbf{ab}}\,\sigma_i^{\mathbf{a}}\sigma_{i+1}^{\mathbf{b}}$, for $\epsilon_{\mathbf{ab}}$ the SL(2;$\mathbb{C}$) invariant tensor on $\mathbb{CP}^1$. (The wavefunctions $\mathscr{A}_i$ are weightless with respect to both the map components and the homogeneous coordinates σ_i of each marked point on $\mathbb{CP}^1$, ensuring that the entire expression is well-defined projectively.) Choosing the $\mathscr{A}_i$ to be minitwistor representatives of (super-)momentum eigenstates, we set

$$\mathscr{A}_i = \int \frac{\mathrm{d}t_i}{t_i}\,\bar{\delta}^2(\lambda_i - t_i\lambda(\sigma_i))\,\exp\left[\mathrm{i}t_i^2\,u(\sigma_i) + \mathrm{i}t_i\eta_{i\,a}\chi^a(\sigma_i)\right]. \tag{4.90}$$

Via the Penrose transform, it is easy to see that such wavefunctions correspond to plane wave momentum eigenstate superfields

$$\mathscr{A}_i \to \exp\mathrm{i}\left(x^{\alpha\beta}\lambda_{i\,\alpha}\lambda_{i\,\beta} + \theta^{a\alpha}\eta_{i\,a}\lambda_{i\,\alpha}\right), \tag{4.91}$$

on space-time, with on-shell supermomenta $\{\lambda_i\lambda_i,\,\lambda_i\eta_i\}$.

First, when $d=0$ the map $Z:\mathbb{CP}^1\to\mathbb{MT}_s$ is in fact constant, so $Z(\sigma)=Z$ and we take $R(\lambda)=1$ as standard. Thus, $\mathrm{d}\mu_C^{(0)} = \mathrm{d}u\,\mathrm{d}^2\lambda\,\mathrm{d}^4\chi/\mathrm{vol}(\mathrm{GL}(2;\mathbb{C}))$, so using the vol $SL(2;\mathbb{C})$ factor to fix $\sigma_1,\sigma_2,\sigma_3$ to three arbitrary points, the three–point $d=0$ amplitude of (4.89) becomes

$$\begin{aligned}\mathscr{M}_{3,0}^{(0)} &= \int \frac{\mathrm{d}u\,\mathrm{d}^2\lambda\,\mathrm{d}^4\chi}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\prod_{i=1}^{3}\frac{(\sigma_i\mathrm{d}\sigma_i)}{(i\,i{+}1)}\,\mathrm{tr}\,(\mathscr{A}_1(\sigma_1)\,\mathscr{A}_2(\sigma_2)\,\mathscr{A}_3(\sigma_3))\\ &= \int \mathrm{d}u\,\langle\lambda\,\mathrm{d}\lambda\rangle\,\mathrm{d}^4\chi\,\mathrm{tr}\,(\mathscr{A}_1\,\mathscr{A}_2\,\mathscr{A}_3)\,.\end{aligned} \tag{4.92}$$

This is just the evaluation of the vertex of the action (4.53) on three on-shell states. We have already demonstrated that this part of the action reduces to the first–order part of the space–time action corresponding to the (super-)Bogomolny equations, so it gives the same amplitudes. In fact, three particle momentum conservation $\lambda_1\lambda_1+\lambda_2\lambda_2+\lambda_3\lambda_3=0$ implies that $\langle 12\rangle$, $\langle 23\rangle$ and $\langle 31\rangle$ all vanish, as also follows from the locality of the vertex in (4.92). Since there are no independent $\tilde{\lambda}_i$s in three dimensions, all three particle amplitudes must vanish even with complexified kinematics.

In fact, all three-point amplitudes vanish in three dimensions as a consequence of momentum conservation (even with complexified kinematics), so the 3-point $\overline{\text{MHV}}$ is zero. However, one can still identify a meaningful $\overline{\text{MHV}}$ pseudo-amplitude as the coefficient of the overall bosonic momentum conserving δ-function. In [72] this pseudo-amplitude was shown to take the form

$$\mathscr{M}_{3,0}^{(0)} = \delta^3\left(\sum_{i=1}^{3}\lambda_i\lambda_i\right)\times\frac{\delta^{0|4}(\eta_1\langle 23\rangle+\eta_2\langle 31\rangle+\eta_3\langle 12\rangle)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} = 0\,, \tag{4.93}$$

It is also straightforward to show that (4.89) reproduces all of the MHV tree amplitudes of $\mathscr{N}=8$ SYMH$_3$. Such amplitudes correspond to degree $d=1$ for the map to $\mathbb{MT}_s$, and the moduli integrations can be performed explicitly against the wavefunctions (4.90), leading to:

$$\mathcal{M}^{(0)}_{n,1} = \delta^3\left(\sum_{i=1}^{n} \lambda_i \lambda_i\right) \delta^8\left(\sum_{i=1}^{n} \lambda_i \eta_i\right) \frac{1}{\langle 12\rangle \langle 23\rangle \cdots \langle n-1\, n\rangle \langle n1\rangle}, \tag{4.94}$$

which is the n-point MHV tree amplitude for $\mathcal{N} = 8$ SYMH$_3$ theory in the form obtained in [72].

The strongest test of the formula's validity is factorization: locality and unitarity of $\mathcal{N} = 8$ SYMH$_3$ dictate that its tree amplitudes should have simple poles on multiparticle factorization channels, with no other singularities. Since (4.89) produces the correct MHV and $\overline{\text{MHV}}$ seed amplitudes, BCFW recursion [54] ensures that it is correct if it factorizes appropriately.

Following studies of factorization for connected formulae of tree-level $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA in $d = 4$ [95, 29], we can probe the factorization behaviour of (4.89) by looking at the limit where the underlying Riemann sphere degenerates. In a standard parametrization, $\Sigma_s \cong \mathbb{CP}^1$ degenerates in the $s \to 0$ limit to two Riemann spheres, Σ_L and Σ_R joined together at a node:

$$\lim_{s\to 0} \Sigma_s = \Sigma_L \cup \Sigma_R\,.$$

If $\sigma^{\mathbf{a}}$ are the homogeneous coordinates on Σ_s, these are related to the natural coordinates on Σ_L and Σ_R by

$$\sigma^{\mathbf{a}}_L = \left(\frac{\sigma^{\mathbf{1}}}{s}, \sigma^{\mathbf{0}}\right), \qquad \sigma^{\mathbf{a}}_R = \left(\frac{\sigma^{\mathbf{0}}}{s}, \sigma^{\mathbf{1}}\right). \tag{4.95}$$

For simplicity, we can work with an affine coordinate z on Σ_s in the coordinate patch where $\sigma^{\mathbf{0}} \neq 0$, leading to

$$z_L = \frac{s}{z}, \qquad z_R = s\,z. \tag{4.96}$$

The origin of the two affine coordinates z_L, z_R is the node $z_\bullet \in \Sigma_L \cap \Sigma_R$ where the two spheres are joined in the $s \to 0$ limit.

We want to determine the behaviour of (4.89) in the $s \to 0$ limit. Without loss of generality, we assume that the n original marked points are distributed into sets L, R of size n_L and n_R on Σ_L and Σ_R, respectively, in this limit. Likewise, the map into minitwistor space will degenerate into maps from Σ_L and Σ_R of degrees d_L and d_R, respectively. These obey

$$n_L + n_R = n, \qquad d_L + d_R = d. \tag{4.97}$$

Straightforward calculations demonstrate the following small s behaviour for the factors in the amplitude which depend only on the coordinates of Σ_s:

$$\frac{\prod_{i=1}^{n} \mathrm{d}z_i}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \cdot \prod_{i=1}^{n} \frac{1}{z_i - z_{i+1}} = s^{-2} \frac{\mathrm{d}s^2}{\prod_{i\in L} z_{Li}^2} \left(\frac{\prod_{j\in L\cup\{\bullet\}} \mathrm{d}z_{Lj}}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \right) \left(\frac{\prod_{k\in R\cup\{\bullet\}} \mathrm{d}z_{Rk}}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \right) \tag{4.98}$$

$$\cdot \prod_{i\in L} z_{Li}^2 \prod_{j\in L\cup\{\bullet\}} \frac{1}{z_{Lj} - z_{Lj+1}} \prod_{k\in R\cup\{\bullet\}} \frac{1}{z_{Rk} - z_{Rk+1}} . \tag{4.99}$$

In particular, the measure and Parke-Taylor factors split into the relevant measures and Parke-Taylor factors on Σ_L and Σ_R, up to overall factors of the parameter s. This is most easily understood in the language of Szego kernels. Define the Szego kernel

$$S(i,j) = \frac{\sqrt{dz_i dz_j}}{z_i - z_j}. \tag{4.100}$$

Then for z_i and z_j both on Σ_L,

$$S(i,j) = \frac{\sqrt{dz_{Li} dz_{Lj}}}{z_{Li} - z_{Lj}} = S_L(i,j), \tag{4.101}$$

and similarly for z_i and z_j both on Σ_R, while for z_i on Σ_L and z_j on Σ_R,

$$S(i,j) = s\frac{\sqrt{dz_{Li} dz_{Rj}}}{z_{Li} z_{Rj}} + \mathcal{O}(s^3) \tag{4.102}$$

$$= s\frac{\sqrt{dz_{Li} dz_{Rj} dz_{L\bullet} dz_{R\bullet}}}{z_{Li} z_{Rj} \sqrt{dz_{L\bullet} dz_{R\bullet}}} + \mathcal{O}(s^3) \tag{4.103}$$

$$= s\frac{\sqrt{dz_{Li} dz_{Rj} dz_{L\bullet} dz_{R\bullet}}}{(z_{Li} - z_{L\bullet})(z_{Rj} - z_{R\bullet}) \sqrt{dz_{L\bullet} dz_{R\bullet}}} + \mathcal{O}(s^3) \tag{4.104}$$

$$= s\frac{S_L(i,\bullet) S_R(\bullet,j)}{\sqrt{dz_{L\bullet} dz_{R\bullet}}} + \mathcal{O}(s^3) \tag{4.105}$$

since in (4.96) we chose the node $z_{L\bullet}$ and $z_{R\bullet}$ to be at 0 in local coordinates. The Parke-Taylor factor becomes

$$\frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_i S(i,i+1) \tag{4.106}$$

$$= \frac{1}{S(a,b)S(b,c)S(c,a)}\left(\frac{s^2}{dz_{L\bullet}\,dz_{R\bullet}}\prod_{j\in L\cup\{\bullet\}} S_L(j,j+1)\prod_{k\in R\cup\{\bullet\}} S_R(k,k+1)+\mathcal{O}(s^4)\right), \tag{4.107}$$

choosing the SL$(2,\mathbb{C})$ to act on the points $a,b\in\Sigma_L$ and $c\in\Sigma_R$,

$$= \frac{s^2}{S_L(a,b)S_L(b,\bullet)S_R(\bullet,c)S_R(c,\bullet)S_L(\bullet,a)ds^2}\frac{ds^2}{s^2}\prod_{j\in L\cup\{\bullet\}} S_L(j,j+1)\prod_{k\in R\cup\{\bullet\}} S_R(k,k+1)+\mathcal{O}(s^2) \tag{4.108}$$

$$= \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})_L\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})_R}\frac{ds^2}{s^2}\prod_{j\in L\cup\{\bullet\}} S_L(j,j+1)\prod_{k\in R\cup\{\bullet\}} S_R(k,k+1)+\mathcal{O}(s^2). \tag{4.109}$$

Orderings which cross between Σ_L and Σ_R more than twice are suppressed by higher powers of s.

Now one must consider the portions of (4.89) which depend on the map to minitwistor space itself. For instance, the λ_α components of the map are written in the affine coordinate on Σ_s as

$$\lambda_\alpha(z) = \sum_{r=0}^{d}\lambda_{\alpha\,r}\,z^r\,. \tag{4.110}$$

Adapting this to the affine coordinate on each branch as $s\to 0$ allows $\lambda_\alpha(z)$ to be rewritten as

$$\begin{aligned}\lambda_\alpha(z) &= z^{d_L}\left(\sum_{r=1}^{d_L}\lambda_{\alpha\,d_L-r}\,\frac{z_L^r}{s^r}+\lambda_{\alpha\bullet}+\sum_{t=1}^{d_R}\lambda_{\alpha\,d_L+t}\,\frac{s^t}{z_L^t}\right) \\ &= z^{d_L}\left(\sum_{r=1}^{d_L}\lambda_{\alpha\,d_L-r}\,\frac{s^r}{z_R^r}+\lambda_{\alpha\bullet}+\sum_{t=1}^{d_R}\lambda_{\alpha\,d_L+t}\,\frac{z_R^t}{s^t}\right),\end{aligned} \tag{4.111}$$

making the identification

$$\lambda_{\alpha\bullet} := \lambda_{\alpha\,d_L}\,. \tag{4.112}$$

Re-defining the map moduli according to

$$\frac{\lambda_{\alpha\,d_L-r}}{s^r}\to\lambda_{\alpha\,r}\,,\qquad \frac{\lambda_{\alpha\,d_L+t}}{s^t}\to\lambda_{\alpha\,t}\,, \tag{4.113}$$

enables us to write the map in a way that is naturally adapted to the degeneration of Σ_s:

$$\lambda_\alpha(z_L) = \lambda_{\alpha\,\bullet} + \sum_{r=1}^{d_L} \lambda_{\alpha\,r}\, z_L^r + \sum_{t=1}^{d_R} \lambda_{\alpha\,t}\, s^{2t}\, z_L^{-t} = \sum_{r=0}^{d_L} \lambda_{\alpha\,r}\, z_L^r + O(s^2)\,, \tag{4.114}$$

and

$$\lambda_\alpha(z_R) = \lambda_{\alpha\,\bullet} + \sum_{r=1}^{d_L} \lambda_{\alpha\,r}\, s^{2r}\, z_R^{-r} + \sum_{t=1}^{d_R} \lambda_{\alpha\,t}\, z_R^t = \sum_{t=0}^{d_R} \lambda_{\alpha\,t}\, z_R^t + O(s^2)\,. \tag{4.115}$$

A similar story holds for the $\chi^a(z)$ map components. The $u(z)$ portion of the map can also be written in a way that manifests the factorization:

$$u(z_L) = u_\bullet + \sum_{r=1}^{2d_L} u_r\, z_L^r + O(s^2) = \sum_{r=0}^{2d_L} u_r\, z_L^r + O(s^2)\,, \tag{4.116}$$

$$u(z_R) = u_\bullet + \sum_{t=1}^{2d_R} u_t\, z_R^t + O(s^2) = \sum_{t=0}^{2d_R} u_t\, z_R^t + O(s^2)\,, \tag{4.117}$$

although this requires a different re-scaling of the map moduli than (4.113):

$$\frac{u_{2d_L-r}}{s^r} \to u_r\,, \qquad u_{2d_L} \to u_\bullet\,, \qquad \frac{u_{2d_L+t}}{s^t} \to u_t\,. \tag{4.118}$$

These choices present the map from Σ_s to minitwistor space in a fashion that manifests factorization into degree d_L and d_R maps in the $s \to 0$ limit.

These rescalings (4.113) for the λ_α and χ^a moduli and (4.118) for the u moduli must be accounted for in the measure on these moduli. It is easy to see that the result is:

$$\mathrm{d}^{2d+1}u\, \mathrm{d}^{2(d+1)|4(d+1)}\lambda \to s^{d_L^2+d_R^2}\, \mathrm{d}u_\bullet\, \mathrm{d}^{2|4}\lambda_\bullet\, \mathrm{d}^{2d_L}u_L\, \mathrm{d}^{2d_L|4d_L}\lambda_L\, \mathrm{d}^{2d_R}u_R\, \mathrm{d}^{2d_R|4d_R}\lambda_R\,. \tag{4.119}$$

The final place where s-scaling can appear in (4.89) is from the resultant $R(\lambda)$ in the denominator. To lowest order in s, one can show that [29]

$$R(\lambda) = s^{d_L^2+d_R^2}\, R(\lambda_L)\, R(\lambda_R)\,, \tag{4.120}$$

where $R(\lambda_L)$, $R(\lambda_R)$ are the resultants of the degree d_L, d_R maps $\lambda_\alpha(z_L)$ and $\lambda_\alpha(z_R)$ from $\Sigma_{L,R}$ to $\mathbb{CP}^1$ which emerge in the $s \to 0$ limit.

Collecting all factors of s, one can now read off the behaviour of the formula as $s \to 0$:

$$\mathcal{M}^{(0)}_{n,d} = \int \frac{\mathrm{d}s^2}{s^2} \frac{\mathrm{d}u_\bullet\, \mathrm{d}^{2|4}\lambda_\bullet}{\mathrm{vol}\,\mathbb{C}^*} \frac{\mathrm{d}^{2d_L}u_L\, \mathrm{d}^{2d_L|4d_L}\lambda_L}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{\mathrm{d}^{2d_R}u_R\, \mathrm{d}^{2d_R|4d_R}\lambda_R}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{1}{R(\lambda_L)R(\lambda_R)}$$
$$\times \prod_{i\in L\cup\{\bullet\}} \frac{\mathrm{d}z_{L\,i}}{z_i - z_{i+1}} \prod_{j\in R\cup\{\bullet\}} \frac{\mathrm{d}z_{R\,j}}{z_j - z_{j+1}} \prod_{k\in L} \mathcal{A}_k \prod_{m\in R} \mathcal{A}_m + O(s^0)\,. \quad (4.121)$$

In particular, the formula features a simple pole in s^2. To see that this corresponds to the simple pole in exchanged momentum, note that the total momentum inserted on Σ_L in the $s \to 0$ limit is

$$P_L^{\alpha\beta} = \sum_{i\in L} \lambda_i^\alpha \lambda_i^\beta\,. \quad (4.122)$$

Now, using the delta functions

$$\prod_{i\in L} \bar{\delta}^2(\lambda_i - t_i \lambda(z_{L\,i}))\,, \quad (4.123)$$

which appear in (4.121) through the wavefunction insertions on Σ_L, it follows that

$$P_L^{\alpha\beta} = \sum_{i\in L} t_i^2\, \lambda^\alpha(z_{L\,i})\, \lambda^\beta(z_{L\,i}) = \sum_{i\in L} t_i^2 \left(\sum_{r=0}^{d_L} \lambda^\alpha_{L\,r}\, z^r_{L\,i}\right)\left(\sum_{t=0}^{d_L} \lambda^\beta_{L\,t}\, z^t_{L\,i}\right) + O(s^2)\,. \quad (4.124)$$

Additionally, performing the $\mathrm{d}^{2d_L+1}u$ moduli integrals leads to a series of delta functions:

$$\prod_{r=0}^{2d_L} \delta\left(\sum_{i\in L} t_i^2 z^r_{L\,i} + O(s^2)\right)\,. \quad (4.125)$$

On the support of these delta functions, the exchanged momentum obeys

$$P_L^{\alpha\beta} = \lambda_\bullet^\alpha \lambda_\bullet^\beta \sum_{i\in L} t_i^2 + O(s^2)\,, \quad (4.126)$$

and therefore

$$P_L^2 = O(s^2)\,. \quad (4.127)$$

Thus, the simple pole in s^2 which appears in (4.121) can be identified with a simple pole of the form P_L^{-2}. So the degeneration limit $s \to 0$ corresponds precisely to the tree-level factorization channel we wanted to probe. Furthermore, the formula (4.89) has the desired simple pole in this channel. It is easy to see that these poles are the only such singularities in the formula, because the resultant $R(\lambda)$ is non-vanishing and all singularities of the Parke-Taylor factor correspond to factorization channels.

To complete the factorization argument, we must account for one set of additional moduli missing from the measure (4.119). This can be done by inserting an auspicious factor of one into the formula:

$$1 = \int\limits_{\mathbb{MT}_s \times \mathbb{MT}_s} \frac{\mathrm{d}u_*\, \mathrm{d}^{2|4}\lambda_*}{\mathrm{vol}\,\mathbb{C}^*} \frac{\mathrm{d}u\, \mathrm{d}^{2|4}\lambda}{\mathrm{vol}\,\mathbb{C}^*} \frac{\mathrm{d}t}{t} \bar{\delta}(u - t^2 u_\bullet)\, \bar{\delta}^{2|4}(\lambda - t\lambda_\bullet)\, \frac{\mathrm{d}r}{r} \bar{\delta}(u - r^2 u_*)\, \bar{\delta}^{2|4}(\lambda - r\lambda_*)\,. \tag{4.128}$$

The measure over $\mathrm{d}u_*\, \mathrm{d}^{2|4}\lambda_*$ (along with its $\mathrm{vol}\,\mathbb{C}^*$ quotient) can now be incorporated into (4.119) to give the full factorized measure, while the new delta functions and scale integrals define state insertions at the node on either side of the factorization channel:

$$\mathscr{A}_{L\bullet} = \int \frac{\mathrm{d}t}{t} \bar{\delta}(u - t^2 u_\bullet)\, \bar{\delta}^{2|4}(\lambda - t\lambda_\bullet)\,, \quad \mathscr{A}_{R\bullet} = \int \frac{\mathrm{d}s}{s} \bar{\delta}(u - s^2 u_*)\, \bar{\delta}^{2|4}(\lambda - s\lambda_*)\,. \tag{4.129}$$

The residue of the formula on the simple pole in exchanged momentum is then

$$\int_{\mathbb{MT}_s} \frac{\mathrm{d}u\, \mathrm{d}^{2|4}\lambda}{\mathrm{vol}\,\mathbb{C}^*} \mathscr{M}^{(0)}_{n_L+1,d_L}\big(\{\lambda_i, \eta_i\}_{i\in L};\, u, \lambda, \chi\big)\, \mathscr{M}^{(0)}_{n_R+1,d_R}\big(u, \lambda, \chi;\, \{\lambda_j, \eta_j\}_{j\in R}\big)\,. \tag{4.130}$$

The remaining integral over minitwistor space is simply the sum over on-shell states in $\mathcal{N} = 8$ SYMH$_3$ flowing through the cut.

4.4.2 Relation to the RSVW formula

We have already remarked on the close similarity between the RSVW formula for tree-level scattering in $\mathcal{N} = 4$ SYM$_4$ and the formula (4.89) for tree-level scattering in $\mathcal{N} = 8$ SYMH$_3$. This similarity is more than heuristic: the $\mathcal{N} = 8$ SYMH$_3$ formula can be viewed as a symmetry reduction of the RSVW formula itself. By expanding (4.80), the n-point N^{d-1}MHV tree amplitude of $\mathcal{N} = 4$ SYM$_4$ is given by

$$\widetilde{\mathscr{M}}^{(0)}_{n,d} = \int \mathrm{d}\tilde{\mu}^{(d)}_{\tilde{C}} \prod_{i=1}^{n} \frac{(\sigma_i \mathrm{d}\sigma_i)}{(i\, i+1)} \operatorname{tr}\Big(\widetilde{\mathscr{A}_1}\, \widetilde{\mathscr{A}_2} \cdots \widetilde{\mathscr{A}_n}\Big)\,, \tag{4.131}$$

where $\widetilde{\mathscr{A}_i}$ are (linearised) insertions of the $\mathcal{N} = 4$ SYM$_4$ on twistor space. Our claim is that $\widetilde{\mathscr{M}}^{(0)}_{n,d}$ is reduced to $\mathscr{M}^{(0)}_{n,d}$ upon replacing the twistor wavefunctions with minitwistor wave functions, $\widetilde{\mathscr{A}_i} \to \mathscr{A}_i$, and taking the symmetry reduction of the measure:

$$\mathscr{T}^{(d)} \lrcorner\, \mathrm{d}\tilde{\mu}^{(d)}_{\tilde{C}} = \frac{\mathrm{d}\mu^{(d)}_{C}}{R(\lambda)}\,. \tag{4.132}$$

This reduction is defined by taking the vector $\mathscr{T}^{(d)}$ on the moduli space of maps from $\mathbb{CP}^1 \to \mathbb{PT}_s$ to be

$$\mathscr{T}^{(d)} := T^{\alpha\dot\alpha}\, \lambda_\alpha^{\mathbf{a}_1\cdots\mathbf{a}_d}\, \frac{\partial}{\partial\mu^{\dot\alpha\,\mathbf{a}_1\cdots\mathbf{a}_d}}\,, \tag{4.133}$$

and the moduli of the u-component of the map $\mathbb{CP}^1 \to \mathbb{MT}_s$ to be

$$u^{\mathbf{a}_1\cdots\mathbf{a}_{2d}} = \lambda_\alpha^{(\mathbf{a}_1\cdots\mathbf{a}_d}\, \mu_{\dot\alpha}^{\mathbf{a}_{d+1}\cdots\mathbf{a}_{2d})}\, T^{\alpha\dot\alpha}\,, \tag{4.134}$$

in terms of the moduli of the map to twistor space, so that $u(\sigma) = [\mu(\sigma)|T|\lambda(\sigma)\rangle$.

The non-trivial part of the claim is the relationship between the measures on the map moduli (4.132). Since both sides of (4.132) are weightless, the scaling weights (with respect to both map moduli and the coordinates on the Riemann sphere) match. Further, it is easy to see that the mass dimensions on both sides of the reduction match. Using the mass dimensions

$$[\lambda] = \frac{1}{2}\,, \quad [\mu] = -\frac{1}{2}\,, \quad [u] = 0\,, \quad [T] = 1\,, \tag{4.135}$$

it is straightforward to see that $[\mathrm{d}\tilde\mu_{\tilde C}^{(d)}] = +1$, while $[\mathrm{d}\mu_C^{(d)}] = \frac{2d+2}{2} = d+1$. The resultant makes up the difference on the minitwistor side, since $[R(\lambda)] = d$.

The relationship (4.132) can also be seen explicitly for low degree maps. When $d = 0$, the relationship is precisely the reduction from $\mathbb{PT}_s$ to $\mathbb{MT}_s$ given by (4.57). At $d = 1$, we can compute directly:

$$\begin{aligned} \mathscr{T}^{(1)} \lrcorner\, \mathrm{d}\tilde\mu_{\tilde C}^{(1)} &= T^{\alpha\dot\alpha}\, \lambda_\alpha^{\mathbf{a}}\, \frac{\partial}{\partial\mu^{\dot\alpha\,\mathbf{a}}} \lrcorner\, \frac{\mathrm{d}^2\mu^{\mathbf{0}} \wedge \mathrm{d}^2\mu^{\mathbf{1}} \wedge \mathrm{d}^{2|4}\lambda^{\mathbf{0}} \wedge \mathrm{d}^{2|4}\lambda^{\mathbf{1}}}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \\ &= \frac{\mathrm{d}u^{\mathbf{00}} \wedge \mathrm{d}u^{\mathbf{01}} \wedge \mathrm{d}u^{\mathbf{11}}}{\langle\lambda^{\mathbf{0}}\,\lambda^{\mathbf{1}}\rangle}\, \frac{\mathrm{d}^{2|4}\lambda^{\mathbf{0}} \wedge \mathrm{d}^{2|4}\lambda^{\mathbf{1}}}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\,, \end{aligned} \tag{4.136}$$

using the identifications (4.133), (4.134). Sure enough, $\langle\lambda^{\mathbf{0}}\,\lambda^{\mathbf{1}}\rangle$ is precisely the resultant for the λ-components of the map when $d = 1$. At $d = 2$ we have $u^{\mathbf{abcd}} = [\mu^{(\mathbf{ab}}|T|\lambda^{\mathbf{cd})}\rangle$ and

$$\mathscr{T}^{(2)} = T^{\alpha\dot\alpha}\left(\lambda_\alpha^{\mathbf{00}}\frac{\partial}{\partial\mu^{\dot\alpha\mathbf{00}}} + \lambda_\alpha^{\mathbf{01}}\frac{\partial}{\partial\mu^{\dot\alpha\mathbf{01}}} + \lambda_\alpha^{\mathbf{11}}\frac{\partial}{\partial\mu^{\dot\alpha\mathbf{11}}}\right). \tag{4.137}$$

Therefore we have

$$\begin{aligned}
\mathrm{d}^5 u = & [\mathrm{d}\mu^{\mathbf{00}}|T|\lambda^{\mathbf{00}}\rangle \wedge \left([\mathrm{d}\mu^{\mathbf{00}}|T|\lambda^{\mathbf{01}}\rangle + [\mathrm{d}\mu^{\mathbf{01}}|T|\lambda^{\mathbf{00}}\rangle\right) \\
& \wedge \left([\mathrm{d}\mu^{\mathbf{00}}|T|\lambda^{\mathbf{11}}\rangle + [\mathrm{d}\mu^{\mathbf{01}}|T|\lambda^{\mathbf{01}}\rangle + [\mathrm{d}\mu^{\mathbf{11}}|T|\lambda^{\mathbf{00}}\rangle\right) \\
& \wedge \left([\mathrm{d}\mu^{\mathbf{01}}|T|\lambda^{\mathbf{11}}\rangle + [\mathrm{d}\mu^{\mathbf{11}}|T|\lambda^{\mathbf{01}}\rangle\right) \wedge [\mathrm{d}\mu^{\mathbf{11}}|T|\lambda^{\mathbf{11}}\rangle \qquad \bmod \mathrm{d}\lambda \\
= & \left(\langle\lambda^{\mathbf{00}}\lambda^{\mathbf{01}}\rangle\langle\lambda^{\mathbf{01}}\lambda^{\mathbf{11}}\rangle - \langle\lambda^{\mathbf{00}}\lambda^{\mathbf{11}}\rangle^2\right) \mathscr{T}^{(2)} \lrcorner \left(\mathrm{d}^2\mu^{\mathbf{00}} \wedge \mathrm{d}^2\mu^{\mathbf{01}} \wedge \mathrm{d}^2\mu^{\mathbf{11}}\right) \qquad \bmod \mathrm{d}\lambda,
\end{aligned} \tag{4.138}$$

where in the first equality we neglect terms which give zero when wedged against $\mathrm{d}^2\lambda^{\mathbf{00}} \wedge \mathrm{d}^2\lambda^{\mathbf{01}} \wedge \mathrm{d}^2\lambda^{\mathbf{11}}$, and the second equality follows by repeated use of the Schouten identity and recalling $T^2 = 1$. The expression $\langle\lambda^{\mathbf{00}}\lambda^{\mathbf{01}}\rangle\langle\lambda^{\mathbf{01}}\lambda^{\mathbf{11}}\rangle - \langle\lambda^{\mathbf{00}}\lambda^{\mathbf{11}}\rangle^2$ is exactly the resultant of the $d = 2$ map, verifying (4.132) in this degree 2 case. Higher degree cases follow similarly.

4.5 Discussion

In this chapter, we have presented a new minitwistor action describing YMH theory in three dimensions, and also its maximally supersymmetric completion. We showed how this action reduces to the standard space–time action. The most obvious question is to understand how to perform perturbation theory using this action, obtaining a Feynman diagram expansion analogous to the MHV diagrams that follow from the twistor action for $\mathcal{N} = 4$ SYM in four dimensions [20, 19, 7]. It would also be interesting to construct an amplitude / super Wilson loop duality in three dimensions. While this might again be expected to mimicking the twistor approach of [76, 26, 5], a significant difference would appear to be that in the three–dimensional case, even non–null separated space-time points correspond to intersecting minitwistor lines. Thus the minitwistor image of a piecewise null polygon appears to have many more 'accidental' self–intersections, whose role in the Wilson loop would need to be understood.

In 4d, this property was closely tied to dual conformal symmetry and the amplitudes of 3d $\mathcal{N} = 8$ SYM were shown to be dual conformal covariant in [72]. Furthermore, dual conformal symmetry was demonstrated for the ABJM theory in [62, 15], and an amplitude/Wilson loop duality was found at 4-points in [32].

We have also presented a connected prescription formula for all tree amplitudes in (supersymmetric) YMH theory, demonstrating its correctness from $\overline{\text{MHV}}$ and MHV examples, and from checking its properties under factorization. As with the RSVW formula in four dimensions, this expression cries out for an understanding in terms of a minitwistor string theory, perhaps along the lines of [35]. A worldsheet model that gauges the action of $T^{\dot{\alpha}\alpha}\lambda_\alpha \partial/\partial\mu^{\dot{\alpha}}$ would seem to be a good starting–point, though it also seems inevitable that any such model will in fact describe YMH theory coupled to some version of Einstein–Weyl gravity, this being the dimensional reduction of self-dual conformal gravity in four dimensions [68].

Perhaps most ambitiously, $3d$ YMH theory is also the arena for Polyakov's beautiful model of confinement through monopole condensation [86]. Given the close relation between monopoles and the local part of the minitwistor action, it would be very interesting to understand how this occurs from the perspective presented here.

References

[1] Adamo, T. (2011). Correlation functions, null polygonal Wilson loops, and local operators. *JHEP*, 12:006.

[2] Adamo, T. (2013). *Twistor actions for gauge theory and gravity.* PhD thesis, University of Oxford.

[3] Adamo, T. (2015). Gravity with a cosmological constant from rational curves. *JHEP*, 11:098.

[4] Adamo, T., Bullimore, M., Mason, L., and Skinner, D. (2011a). A Proof of the Supersymmetric Correlation Function / Wilson Loop Correspondence. *JHEP*, 08:076.

[5] Adamo, T., Bullimore, M., Mason, L., and Skinner, D. (2011b). Scattering Amplitudes and Wilson Loops in Twistor Space. *J. Phys.*, A44:454008.

[6] Adamo, T. and Groechenig, M. (2013). Moduli stacks of maps for supermanifolds. *Adv. Theor. Math. Phys.*, 17(6):1303–1342.

[7] Adamo, T. and Mason, L. (2012). MHV diagrams in twistor space and the twistor action. *Phys. Rev.*, D86:065019.

[8] Adamo, T., Skinner, D., and Williams, J. (2016). Twistor methods for AdS_5. *JHEP*, 08:167.

[9] Adamo, T., Skinner, D., and Williams, J. (2018). Minitwistors and 3d Yang-Mills-Higgs theory. *J. Math. Phys.*, 59(12):122301.

[10] Agarwal, A. and Young, D. (2011). Manifest SO(N) invariance and S-matrices of three-dimensional N=2,4,8 SYM. *JHEP*, 05:100.

[11] Alday, L. F., Maldacena, J., Sever, A., and Vieira, P. (2010). Y-system for Scattering Amplitudes. *J. Phys.*, A43:485401.

[12] Atiyah, M., Dunajski, M., and Mason, L. J. (2017). Twistor theory at fifty: from contour integrals to twistor strings. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 473(2206):20170530.

[13] Bailey, T., Eastwood, M., and Gover, A. (1994). Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217.

[14] Bailey, T. N. and Dunne, E. G. (1998). A twistor correspondence and Penrose transform for odd-dimensional hyperbolic space. *Proc.Amer.Math.Soc.*, 126:1245–1252.

[15] Bargheer, T., Loebbert, F., and Meneghelli, C. (2010). Symmetries of Tree-level Scattering Amplitudes in N=6 Superconformal Chern-Simons Theory. *Phys. Rev.*, D82:045016.

[16] Baston, R. J. (1990). Verma modules and differential conformal invariants. *J. Differential Geom.*, 32(3):851–898.

[17] Baston, R. J. and Eastwood, M. G. (1989). *The Penrose transform, its interaction with representation theory.* Oxford University Press.

[18] Birrell, N. D. and Davies, P. C. W. (1982). *Quantum field theory in curved spacetime*, page 36â€“88. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

[19] Boels, R., Mason, L. J., and Skinner, D. (2007a). From Twistor Actions to MHV Diagrams. *Phys. Lett.*, B648:90–96.

[20] Boels, R., Mason, L. J., and Skinner, D. (2007b). Supersymmetric Gauge Theories in Twistor Space. *JHEP*, 02:014.

[21] Bonezzi, R., Latini, E., and Waldron, A. (2010). Gravity, Two Times, Tractors, Weyl Invariance and Six Dimensional Quantum Mechanics. *Phys. Rev.*, D82:064037.

[22] Braden, H. W. and Enolski, V. Z. (2018). The Construction of Monopoles. *Commun. Math. Phys.*, 362(2):547–570.

[23] Branson, T. P. (1985). Differential operators cononically associated to a conformal structure. *MATHEMATICA SCANDINAVICA*, 57:293–345.

[24] Britto, R., Cachazo, F., and Feng, B. (2005a). New recursion relations for tree amplitudes of gluons. *Nucl. Phys.*, B715:499–522.

[25] Britto, R., Cachazo, F., Feng, B., and Witten, E. (2005b). Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys. Rev. Lett.*, 94:181602.

[26] Bullimore, M. and Skinner, D. (2011). Holomorphic Linking, Loop Equations and Scattering Amplitudes in Twistor Space.

[27] Cachazo, F. (2013). Resultants and Gravity Amplitudes.

[28] Cachazo, F., He, S., and Yuan, E. Y. (2013). Scattering in Three Dimensions from Rational Maps. *JHEP*, 10:141.

[29] Cachazo, F., Mason, L., and Skinner, D. (2014). Gravity in Twistor Space and its Grassmannian Formulation. *SIGMA*, 10:051.

[30] Cachazo, F., Svrcek, P., and Witten, E. (2004). MHV vertices and tree amplitudes in gauge theory. *JHEP*, 09:006.

[31] Chalmers, G. and Siegel, W. (1996). The Selfdual sector of QCD amplitudes. *Phys. Rev.*, D54:7628–7633.

[32] Chen, W.-M. and Huang, Y.-t. (2011). Dualities for Loop Amplitudes of N=6 Chern-Simons Matter Theory. *JHEP*, 11:057.

[33] Chicherin, D., Doobary, R., Eden, B., Heslop, P., Korchemsky, G. P., Mason, L., and Sokatchev, E. (2015). Correlation functions of the chiral stress-tensor multiplet in $\mathcal{N} = 4$ SYM. *JHEP*, 06:198.

[34] Chicherin, D. and Sokatchev, E. (2016). Demystifying the twistor construction of composite operators in N=4 super-Yang-Mills theory.

[35] Chiou, D.-W., Ganor, O. J., Hong, Y. P., Kim, B. S., and Mitra, I. (2005). Massless and massive three dimensional super Yang-Mills theory and mini-twistor string theory. *Phys. Rev.*, D71:125016.

[36] Corrigan, E. and Goddard, P. (1981). An n Monopole Solution With 4n-1 Degrees of Freedom. *Commun. Math. Phys.*, 80:575.

[37] Costa, M., Goncalves, V., and Penedones, J. (2014). Spinning ads propagators. *JHEP*, 09:064.

[38] Curry, S. and Gover, A. R. (2014). An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity.

[39] D'Hoker, E. and Freedman, D. Z. (2002). Supersymmetric gauge theories and the AdS / CFT correspondence. In *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pages 3–158.

[40] Dirac, P. (1964). *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, monograph series. Belfer Graduate School of Science, Yeshiva University.

[41] Dirac, P. A. M. (1930). *The Principles of Quantum Mechanics.* Oxford University Press.

[42] Dirac, P. A. M. (1935). The electron wave equation in de-sitter space. *Annals of Mathematics*, 36(3):657–669.

[43] Dirac, P. A. M. (1936). Wave equations in conformal space. *Annals Math.*, 37:429–442.

[44] Dirac, P. A. M. (1949). Forms of relativistic dynamics. *Rev. Mod. Phys.*, 21:392–399.

[45] Dirac, P. A. M. (1950). Generalized Hamiltonian dynamics. *Can. J. Math.*, 2:129–148.

[46] Dunajski, M. (2010). *Solitons, instantons, and twistors.*

[47] Eastwood, M. (2014). Wave equations in conformal space. *Bull. Amer. Math. Soc.*, 51:499–503.

[48] Eastwood, M. G., Penrose, R., and Wells, R. O. (1981). Cohomology and massless fields. *Commun.Math.Phys.*, 78:305–351.

[49] Eastwood, M. G. and Rice, J. W. (1987). Conformally invariant differential operators on minkowski space and their curved analogues. *Comm. Math. Phys.*, 109(2):207–228.

[50] Fefferman, C. and Graham, C. R. (1985). Conformal invariants. In *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, number S131 in Astérisque, pages 95–116. Société mathématique de France.

[51] Fefferman, C. and Graham, C. R. (2011). The ambient metric. *Ann. Math. Stud.*, 178:1–128.

[52] Fitzpatrick, A. L., Kaplan, J., Khandker, Z., Li, D., Poland, D., and Simmons-Duffin, D. (2014). Covariant approaches to superconformal blocks. *JHEP*, 08:129.

[53] Fitzpatrick, A. L., Kaplan, J., Penedones, J., Raju, S., and van Rees, B. C. (2011). A Natural Language for AdS/CFT Correlators. *JHEP*, 11:095.

[54] Gang, D., Huang, Y.-t., Koh, E., Lee, S., and Lipstein, A. E. (2011). Tree-level Recursion Relation and Dual Superconformal Symmetry of the ABJM Theory. *JHEP*, 03:116.

[55] Goldberger, W., Skiba, W., and Son, M. (2012). Superembedding methods for 4d $\mathcal{N} = 1$ scfts. *Phys. Rev.*, D86:025019.

[56] Graham, C. R. (2000). Volume and area renormalizations for conformally compact Einstein metrics. *Rend. Circ. Mat. Palermo*, S63:31–42.

[57] Gubser, S. S., Klebanov, I. R., and Polyakov, A. M. (1998). Gauge theory correlators from noncritical string theory. *Phys. Lett.*, B428:105–114.

[58] Henningson, M. and Sfetsos, K. (1998). Spinors and the AdS / CFT correspondence. *Phys. Lett.*, B431:63–68.

[59] Hitchin, N. J. (1982). Monopoles and geodesics. *Commun. Math. Phys.*, 83:579–602.

[60] Hitchin, N. J. (1983). On the Construction of Monopoles. *Commun. Math. Phys.*, 89:145–190.

[61] Huang, Y.-t. and Lipstein, A. E. (2010a). Amplitudes of 3D and 6D Maximal Superconformal Theories in Supertwistor Space. *JHEP*, 10:007.

[62] Huang, Y.-t. and Lipstein, A. E. (2010b). Dual Superconformal Symmetry of N=6 Chern-Simons Theory. *JHEP*, 11:076.

[63] Huggett, S. A. and Tod, K. P. (1994). *An Introduction to Twistor Theory*. London Mathematical Society Student Texts. Cambridge University Press, 2 edition.

[64] Hughston, L. P. and Hurd, T. R. (1983). A CP**5 Calculus for Space-time fields. *Phys. Rept.*, 100:275–326.

[65] Hughston, L. P. and Shaw, W. T. (1987). Minimal Curves in Six-dimensions. *Class. Quant. Grav.*, 4:869.

[66] Isenberg, J., Yasskin, P. B., and Green, P. S. (1978). Nonselfdual Gauge Fields. *Phys. Lett.*, B78:462–464.

[67] Jones, P. E. (1984). *Minitwistors*. PhD thesis, University of Oxford.

[68] Jones, P. E. and Tod, K. P. (1985). Minitwistor spaces and Einstein-Weyl spaces. *Class. Quant. Grav.*, 2(4):565–577.

[69] Klauder, J. R. (1996). Coherent state quantization of constraint systems. In *Current topics in physics. Proceedings, Inauguration Conference of the Asia-Pacific Center for Theoretical Physics (APCTP), Seoul, Korea, June 4-10, 1996. Vol. 1, 2*, pages 796–833.

[70] Koster, L., Mitev, V., Staudacher, M., and Wilhelm, M. (2016). Composite Operators in the Twistor Formulation of $\mathcal{N} = 4$ SYM Theory. *Phys. Rev. Lett.*, 117:011601.

[71] LeBrun, C. (1983). Spaces of complex null geodesics in complex Riemannian geometry. *Trans.Amer.Math.Soc.*, 278:209–231.

[72] Lipstein, A. E. and Mason, L. (2013). Amplitudes of 3d Yang Mills Theory. *JHEP*, 01:009.

[73] Maldacena, J. M. (1999). The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133. [Adv. Theor. Math. Phys.2,231(1998)].

[74] Mason, L. J. (2005). Twistor actions for non-self-dual fields: A Derivation of twistor-string theory. *JHEP*, 10:009.

[75] Mason, L. J., Reid-Edwards, R. A., and Taghavi-Chabert, A. (2012). Conformal Field Theories in Six-Dimensional Twistor Space. *J.Geom.Phys.*, 62:2353–2375.

[76] Mason, L. J. and Skinner, D. (2010). The Complete Planar S-matrix of N=4 SYM as a Wilson Loop in Twistor Space. *JHEP*, 12:018.

[77] Movshev, M. (2006). On the Berezinian of a moduli space of curves in $\mathbb{P}^{n|n+1}$.

[78] Mueck, W. and Viswanathan, K. S. (1998). Conformal field theory correlators from classical field theory on anti-de Sitter space. 2. Vector and spinor fields. *Phys. Rev.*, D58:106006.

[79] Murray, M. K. (1984). Nonabelian Magnetic Monopoles. *Commun. Math. Phys.*, 96:539–565.

[80] Nahm, W. (1980). A Simple Formalism for the BPS Monopole. *Phys. Lett.*, 90B:413–414.

[81] Paulos, M. F. (2011). Towards Feynman rules for Mellin amplitudes. *JHEP*, 10:074.

[82] Penedones, J. (2011). Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 03:025.

[83] Penrose, R. (1968). Twistor quantisation and curved space-time. *International Journal of Theoretical Physics*, 1(1):61–99.

[84] Penrose, R. (1969). Solutions of the zero-rest-mass equations. *J.Math.Phys.*, 10:38–39.

[85] Penrose, R. and Rindler, W. (1986). *Spinors and Space-Time*, volume 2. Cambridge University Press.

[86] Polyakov, A. M. (1977). Quark Confinement and Topology of Gauge Groups. *Nucl. Phys.*, B120:429–458.

[87] Popov, A. D., Saemann, C., and Wolf, M. (2005). The Topological B-model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations. *JHEP*, 10:058.

[88] Roiban, R. and Siegel, W. (2000). Superstrings on AdS(5) x S**5 supertwistor space. *JHEP*, 11:024.

[89] Roiban, R., Spradlin, M., and Volovich, A. (2004). On the tree level S matrix of Yang-Mills theory. *Phys. Rev.*, D70:026009.

[90] Saemann, C. and Wolf, M. (2013). On Twistors and Conformal Field Theories from Six Dimensions. *J. Math. Phys.*, 54:013507.

[91] Shabanov, S. (1989). Path integral in holomorphic representation without gauge fixation. *Dubna preprint E2-89-687*.

[92] Simmons-Duffin, D. (2014). Projectors, Shadows, and Conformal Blocks. *JHEP*, 04:146.

[93] Sinkovics, A. and Verlinde, E. P. (2005). A Six dimensional view on twistors. *Phys. Lett.*, B608:142–150.

[94] Skenderis, K. (2002). Lecture notes on holographic renormalization. *Class. Quant. Grav.*, 19:5849–5876.

[95] Skinner, D. (2011). A Direct Proof of BCFW Recursion for Twistor-Strings. *JHEP*, 01:072.

[96] Skinner, D. (2013). Twistor Strings for N=8 Supergravity.

[97] Szekeres, P. and Bondi, H. (1968). Conformal tensors. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 304(1476):113–122.

[98] Thomas, T. Y. (1925). Announcement of a projective theory of affinely connected manifolds. *Proceedings of the National Academy of Sciences*, 11(10):588–589.

[99] Thomas, T. Y. (1926). On conformal geometry. *Proceedings of the National Academy of Sciences*, 12(5):352–359.

[100] Thomas, T. Y. (1934). *The differential invariants of generalized spaces*. Cambridge University Press.

[101] Tsai, C. (1996). *The Penrose transform for Einstein-Weyl and Related Spaces*. PhD thesis, University of Edinburgh.

[102] Veblen, O. (1935). A conformal wave equation. *Proceedings of the National Academy of Sciences*, 21(7):484–487.

[103] Vergu, C. (2007). On the Factorisation of the Connected Prescription for Yang-Mills Amplitudes. *Phys. Rev.*, D75:025028.

[104] Ward, R. (1998). Twistors, geometry and integrable systems. In S. A. Huggett, Lionel J. Mason, K. P. T. S. T. and Woodhouse, N. M. J., editors, *The Geometric Universe: Science, Geometry, and the Work of Roger Penrose*, chapter 6, pages 99–108. Oxford University Press, Oxford.

[105] Ward, R. S. (1977). On Selfdual gauge fields. *Phys. Lett.*, A61:81–82.

[106] Ward, R. S. (1981). A Yang-Mills Higgs Monopole of Charge 2. *Commun. Math. Phys.*, 79:317–325.

[107] Ward, R. S. (1985). Integrable and solvable systems, and relations among them. *Phil. Trans. Roy. Soc. Lond.*, A315:451–457.

[108] Ward, R. S. (1988). Soliton solutions in an integrable chiral model in 2+1 dimensions. *Journal of Mathematical Physics*, 29(2):386–389.

[109] Ward, R. S. (1989). Twistors in 2+1 dimensions. *J. Math. Phys.*, 30(10):2246.

[110] Ward, R. S. and Wells, R. O. (1990). *Twistor Geometry and Field Theory*. Cambridge University Press.

[111] Weinberg, S. (2010). Six-dimensional Methods for Four-dimensional Conformal Field Theories. *Phys. Rev.*, D82:045031.

[112] Witten, E. (1978). An Interpretation of Classical Yang-Mills Theory. *Phys. Lett.*, B77:394–398.

[113] Witten, E. (1986). Twistor - Like Transform in Ten-Dimensions. *Nucl. Phys.*, B266:245.

[114] Witten, E. (1998). Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291.

[115] Witten, E. (2004). Perturbative gauge theory as a string theory in twistor space. *Commun. Math. Phys.*, 252:189–258.

[116] Woodhouse, N. M. J. (1985). Real methods in twistor theory. *Class. Quant. Grav.*, 2:257–291.

[117] Wünsch, V. (1986). On conformally invariant differential operators. *Mathematische Nachrichten*, 129(1):269–281.

Appendix A

AdS Laplacian and Dirac equation in embedding space coordinates

The AdS_5 metric,

$$g_{IJ} = -\frac{\eta_{IJ}}{X^2} + \frac{X_I X_J}{X^4} \tag{A.1}$$

was chosen to be basic, i.e. to contract to zero with the Euler vector field $X \cdot \partial$ generating scalings on $\mathbb{CP}_5$. We choose our inverse metric in the same way

$$g^{IJ} = -X^2 \eta^{IJ} + X^I X^J \tag{A.2}$$

where X_I is raised to X^I using η^{IJ}. In order to calculate the Laplacian, we deform away from this degenerate inverse metric using a parameter ϵ

$$g_\epsilon^{IJ} = -X^2 \eta^{IJ} + (1-\epsilon^2) X^I X^J. \tag{A.3}$$

A calculation reveals that $\det(g^{-1}) = \epsilon^2 X^{12}$. We compute the Laplacian acting on $|X|^{-\Delta}\Phi$, which has weight $-\Delta$ because Φ is homogeneous. It is simple to show that

$$\begin{aligned}
\Box_{\mathrm{AdS}} &= \epsilon X^6 \partial_I \left(\frac{1}{\epsilon X^6} \left(-X^2 \eta^{IJ} + (1-\epsilon^2) X^I X^J \right) \partial_J \right) && \text{(A.4)} \\
&= -X^2 \partial^2 - 4\Delta + (1-\epsilon^2)\Delta^2 && \text{(A.5)} \\
&= -X^2 \partial^2 + \Delta(\Delta - 4) - \epsilon^2 \Delta^2, && \text{(A.6)}
\end{aligned}$$

which reduces the the desired expression in the limit $\epsilon \to 0$.

The Dirac equation works similarly. Suitable vierbeins are

$$e_I^P = \frac{\delta_I^K}{|X|} - \left(1 - \frac{1}{\epsilon}\right)\frac{X_I X^P}{|X|^3} \tag{A.7}$$

$$e_P^I = |X|\delta_P^I - (1-\epsilon)\frac{X^I X_P}{|X|} \tag{A.8}$$

The spin connection is

$$\omega_I^{PQ} = \frac{1}{2}e^{JP}\left(\partial_I e_J^Q - \partial_J e_I^Q\right) - \frac{1}{2}e^{JQ}\left(\partial_I e_J^P - \partial_J e_I^P\right) - \frac{1}{2}e^{JP}e^{KQ}\left(\partial_J e_{KR} - \partial_K e_{JR}\right)e_I^R \tag{A.9}$$

$$= \frac{1}{X^2}\left(X^P\delta_I^Q - X^Q\delta_I^P\right) \tag{A.10}$$

The Dirac equation takes the simple form

$$\not{D}_{\text{AdS}}^{AB} = |X|\partial^{AB} + \frac{X^{AB}}{|X|}\left(\frac{5}{2} + X\cdot\partial - \epsilon X\cdot\partial\right). \tag{A.11}$$

Acting on spinors of the form $|X|^{-\Delta-\frac{1}{2}}\Psi$ where Ψ is homogeneous and taking the limit $\epsilon \to 0$, this reduces to

$$\not{D}_{\text{AdS}}^{AB} = |X|\partial^{AB} - (\Delta - 2)\frac{X^{AB}}{|X|}. \tag{A.12}$$